Topology Qualifying exam, Fall 2006

You have three hours to answer these questions. No notes or books are allowed. All the best.

1. Construct a 2 dimensional connected CW complex X with one 0-cell and one 2-cell, whose fundamental group has the presentation:

   $$\pi_1(X) = \langle a, b, c \mid abca = cb \rangle$$

   You may express X as the identification space of a polygon.

2. Give an example of a space X such that $H^i(X, \mathbb{Z}) = \mathbb{Z}$ for all $0 \leq i \leq \infty$, and such that the cohomology ring $H^*(X, \mathbb{Z})$ is finitely generated.

3. How many connected covering spaces does $\mathbb{RP}^3 \times \mathbb{RP}^7$ have? Can you identify any of them?

4. Assume that $\mathbb{RP}^n$ can be covered by k contractible closed subsets. Prove that $k > n$.

   Hint: Use the mod 2 cohomology ring structure of $\mathbb{RP}^n$ and the fact that the degree 1 generator restricts to zero on any contractible subset.

5. Let M be a $2n$ dimensional compact manifold without boundary. Show that

   $$\dim H^n(M, \mathbb{Z}/2) = \chi(M) \bmod 2$$

   where $\chi(M)$ denotes the Euler characteristic of M.

6. Let k be an even integer, and let n by any arbitrary positive integer, show that there is a map $\varphi : S^{2n+1} \longrightarrow \mathbb{RP}^{2n+1}$ of degree k. Show that there is no map from S^{2n+1} to $\mathbb{RP}^{2n+1}$ of odd degree.

7. Let M be a 4-dimensional compact, connected, simply connected manifold without boundary such that $\chi(M) = k$. Assuming M is orientable, calculate $H_i(M, \mathbb{Z})$ for $0 \leq i \leq 4$.

8. Let X be a connected space, show that the suspension of X, ΣX is simply connected. Can we drop the connectedness assumption?