Distance 2 Permutation Statistics and Symmetric Functions

Christopher Severs
Advisor: Professor Jeffrey Remmel, UCSD

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1 Introduction

This paper is presented as an undergraduate honors thesis by Christopher Severs under the supervision of Professor Jeffrey Remmel at UCSD. The aim of the project was to read material about permutation statistics and generating functions for the ring of symmetric functions and then address a problem not covered in the literature to date. In working on this project the author gained a much better understanding of this particular area of mathematics and some insight into how the process of research and writing works.

The main goal of this paper is find generating functions for some new permutation statistics on certain subsets of the symmetric group S_n by defining an appropriate homomorphism of the ring of symmetric functions and then applying that homomorphism to a simple symmetric function identity. This idea was first introduced by Brenti [4].

In [4], Brenti introduces a homomorphism from the ring of symmetric functions to polynomials in a single variable that demonstrates a remarkable connection between permutation enumeration and symmetric functions. Specifically, if Λ is the ring of symmetric functions and e_k is the kth elementary symmetric function, Brenti defines $\xi : \Lambda \to Q[x]$ by

$$\xi(e_k) = \frac{(x-1)^{k-1}}{k!}$$

where $\xi(e_0) = 1$. Now let h_k be the kth complete homogenous symmetric function and p_k the kth power sum symmetric function. Also, for a permutation σ in the symmetric group S_n, let $\mathrm{des}(\sigma)$ and $\mathrm{exc}(\sigma)$ denote the number of descents and excedances of σ, respectively. Then Brenti shows

$$n!\xi(h_n) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)}$$

$$\frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{\sigma \in S_n(\lambda)} x^{\mathrm{exc}(\sigma)} \tag{1}$$

where if $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ is a partition of n, then $S_n(\lambda)$ is the set of permutations in S_n with cycle type λ, and $z_\lambda = \prod_{i=1}^n i^{m_i} m_i!$.

Brenti's proofs of are mainly algebraic. In [2], Beck and Remmel give combinatorial proofs that allow them to give interesting q-analogues. Beck and Remmel define a homomorphism $\xi_q : \Lambda \to Q(q)[x]$ by

$$\xi_q(e_k) = \frac{(x-1)^{k-1} q^{\binom{k}{2}}}{[k]!}$$

where for a positive integer k, $[k] = 1 + q + \cdots + q^{k-1}$ and $[k]! = [k][k-1]\cdots[1]$. They prove that

$$\begin{aligned} [n]!\xi_q(h_n) &= \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)} \\ [n]!\xi_q(p_n) &= \sum_{\sigma \in S_n} x^{\mathrm{rise}(\sigma) - f(\sigma) + 1} q^{\mathrm{coinv}(\sigma)} \left(x^{f(\sigma)} - (x-1)^{f(\sigma)} \right) \end{aligned} \tag{2}$$

where $\mathrm{rise}(\sigma)$ and $\mathrm{coinv}(\sigma)$ are the number of rises and coinversions of σ, respectively, and $f(\sigma)$ is the length of the last increasing sequence of σ when σ is written in one-line notation.

It is this combinatorial approach of Beck and Remmel that we will exploit in this paper. We should note that there has been a whole series of papers that that have used this approach to find generating functions for various permuations statistics on the symmetric group, the hyperoctahedaral group B_n, and various wreath products of the form $G \wr S_n$. These include

- Beck's thesis together with two follow-up papers, one of which was the paper co-authored with Remmel described in detail in the previous section [1, 2, 3],
- a paper by Ram, Remmel, and Whitehead [12],
- Wagner's thesis together with a follow-up paper [13, 14],
- Langley's thesis together with a follow-up paper, [7, 8], and
- a paper by Langley and Remmel [9].

In Section 2 we will present the necessary background material required to arrive at the new Distance 2 results. In this section we also give an example of a combinatorial proof from Mendes and Remmel [11] which the proofs in Section 3 are modeled after.

Section 3 contains the new definitions for the Distance 2 statistics as well as new identities and the generating functions that follow from those identities. We give combinatorial proofs of the identities as well as some q-analogues and n-tuple cases.

2 Background

In this section we present the machinery needed to talk about the results obtained during the course of the project. As such the majority of this section is borrowed from a work by Prof. Remmel and Tony Mendes [11], which was used as a primer on the material covered during the project.

2.1 Symmetric Functions

We must begin with some background information on symmetric functions. Let S_n be the symmetric group. This group is the set of all permutations on $\{1, 2, \ldots, n\}$ where each σ in S_n is a mapping, $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. We can write a permutation $\sigma \in S_n$ in three different ways:

1. Two line notation, where we see where each integer from $1, \ldots, n$ is mapped

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 \end{matrix}$$

   In this example we see that $\sigma(1) = 4$, $\sigma(2) = 1$, etc.

2. We can also write the above permutation just in one line notation

$$4\ 1\ 2\ 3\ 5$$

   In this case the top line from the two line notation is omitted but we still interpret this one line notation in the same fashion.

3. We can also write the permutation in cycle notation

$$(1, 4, 3, 2)(5)$$

   If we start with 1 we see that 1 goes to 4. Then 2 goes to 1, etc.

Let $\sigma \in S_n$ and $P(x_1, \ldots, x_n)$ be a polynomial on n variables. Then $\sigma P(x_1, \cdots, x_n) = P(x_{\sigma_1}, \ldots, x_{\sigma_n})$. We call P symmetric if and only if, $\forall \sigma \in S_n$, $\sigma P(x_1, \cdots, x_n) = P(x_{\sigma_1}, \ldots, x_{\sigma_n}) = P(x_1, \ldots, x_n)$. What this means is that we can permute the variables $x_1, \ldots, x_n$ and not change P itself.
Two examples:

1. $x_1x_2^2 + x_2x_1^2 + x_3$ is not symmetric since if we apply $(1, 3)$ to the indices we get a different function.

2. $x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2$ is symmetric.

Let Λ^N be the ring of symmetric polynomials in $x_1, \ldots, x_N$ and Λ_n^N be the subset of Λ^N containing the homogeneous elements of degree n. Using the surjective ring homomorphism from Λ_n^{N+1} to Λ_n^N defined by taking $x_{N+1} = 0$, let $\Lambda_n = \varprojlim_N \Lambda_n^N$ for each $n \geq 0$. Define $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ to be the ring of symmetric functions. This technical definition of the ring of symmetric functions is needed to ensure the validity of taking an infinite series of monomials in an infinite number of variables. For $\lambda \vdash n$, the monomial symmetric function m_λ is the element in Λ_n given by the sum of all monomials where the exponents on the powers of x_i give a rearrangement of the parts of λ. For example, $m_{(2,1)}$ in 3 variables (meaning that $x_4 = x_5 = \cdots = 0$) is given below:

$$m_{(1,2)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2.$$

It is not difficult to see that any symmetric function where every term is of degree n must be a sum of monomial symmetric functions; therefore, $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n. This implies that the dimension of Λ_n is the number of partitions of n.

The elementary symmetric function e_n may be defined by using a formal power series in $\Lambda[[t]]$. Let

$$\sum_{n=0}^{\infty} e_n t^n = \prod_i (1 + x_i t). \tag{3}$$

Let $E(t)$ be the sum on the left hand side of the above equation. Since only one power of x_i may contribute to the coefficient of t^n on the left hand side of (3) for every i, e_n is the sum of all square free monomials in the variables $x_1, \ldots, x_N$. For example,

$$e_3(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4.$$

The homogeneous symmetric function h_n is defined such that

$$\sum_{n=0}^{\infty} h_n t^n = \prod_i \frac{1}{1 - x_i t}. \tag{4}$$

Let $H(t)$ be the sum on the left hand side of the above equation. For example, h_3 in 3 variables is given below:

$$h_3(x_1, x_2, x_3) = x_1 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 + x_1^3 + x_2^3 + x_3^3.$$

The definitions of the homogeneous and elementary symmetric functions give

$$\sum_{n=0}^{\infty} h_n t^n = H(t) = (E(-t))^{-1} = \left(\sum_{n=0}^{\infty} e_n (-t)^n \right)^{-1}. \tag{5}$$

It is primarily this identity that we will use to construct our new generating functions.

Another basic fact as a result of these definitions is Lemma 1 below.

Lemma 1. *For $n \geq 1$,*

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.$$

Proof 1. Compare the coefficient of t^n on both sides of

$$1 = H(t)E(-t) = \left(\sum_{n=0}^{\infty} h_n t^n\right)\left(\sum_{i=0}^{\infty}(-1)^i e_i t^i\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^i e_i h_{n-i}\right) t^n.$$

□

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, i.e. if $n = \sum_{i=1}^{k} \lambda_i$, then we write $\lambda \vdash n$ and we let $\ell(\lambda) = k$, the number of parts of λ. We also define

$$\begin{aligned} h_\lambda &= h_{\lambda_1} \cdots h_{\lambda_k} \text{ and} \\ e_\lambda &= e_{\lambda_1} \cdots e_{\lambda_k}. \end{aligned}$$

This given, we have the following well known result, see [10].

Theorem 2. *1. $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

2. $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.

3. $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.

4. $e_0, e_1, \ldots$ are algebraically independent and they generate Λ.

Given two bases $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ of Λ_n, we define $M(a,b)_{\lambda,\mu}$, for all partitions λ and μ of n, by

$$b_\lambda = \sum_{\mu \vdash n} a_\mu M(a,b)_{\mu,\lambda}. \tag{6}$$

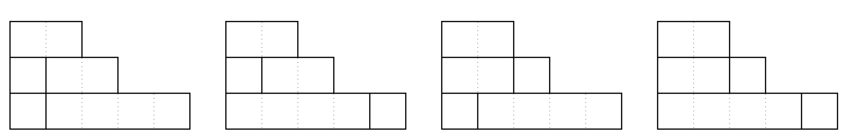
Thus if $\langle a_\lambda \rangle_{\lambda \vdash n}$ and $\langle b_\lambda \rangle_{\lambda \vdash n}$ as row vectors, then the matrix $M(a,b) = ||M(a,b)_{\lambda,\mu}||$ is the transition matrix between that bases $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$, that is,

$$\langle b_\lambda \rangle_{\lambda \vdash n} = \langle a_\lambda \rangle_{\lambda \vdash n} M(a,b). \tag{7}$$

An important ingredient in our results will be an explicit combinatorial interpretation of the elements of the transition matrix $M(e,h)$ between the basis $\{e_\lambda : \lambda \vdash n\}$ and $\{h_\lambda : \lambda \vdash n\}$. That is, we want a combinatorial interpretation for $M(e,h)_{\lambda,\mu}$ where

$$h_\lambda = \sum_{\mu \vdash n} e_\mu M(a,b)_{\mu,\lambda}. \tag{8}$$

Given two partitions λ and μ, let us define a object known as a brick tabloid of shape μ and type λ. The set of all such objects will be denoted by $B_{\lambda,\mu}$. A $T \in B_{\lambda,\mu}$ is formed by partitioning the rows of the Ferrers diagram of λ into "bricks" such that the lengths of the bricks induce the partition μ. For example, we now show all possible brick tabloids of shape $(2,3,5)$ and type $(1^2, 2^2, 4)$:



Theorem 3. *For $\mu \vdash n$,*

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda.$$

Proof. To unclutter notation, let $M(e,h)_{\lambda,\mu} = M_{\lambda,\mu}$ be the coefficient of e_λ in h_μ for the remainder of this proof. If $\lambda \vdash n$, let $\lambda \setminus i$ be the partition λ with a part of size i removed. In the case where λ does not have a part of this size, $\lambda \setminus i$ is undefined and $M_{\lambda\setminus i,\mu} = 0$ by convention.

First, we will show that the numbers $M_{\lambda,\mu}$ satisfy the following:

1. $M_{(n),(n)} = (-1)^{n-1}$,
2. $M_{\lambda,(n)} = \sum_{i=1}^{n-1} (-1)^{i-1} M_{\lambda\setminus i,(n-i)}$ for λ a partition of n with more than one part, and
3. $M_{\lambda,\mu} = \sum M_{\alpha,(\mu_1)} M_{\beta,\mu\setminus\mu_1}$ where the sum runs over all possible partitions $\alpha \vdash \mu_1$ and $\beta \vdash n - \mu_1$ such that the multiset union of the parts of α and β is equal to λ (written $\alpha + \beta = \lambda$) and μ is a partition of n with more than one part.

Lemma 1 may be rewritten to read

$$h_n = (-1)^{n-1} e_n + \sum_{i=1}^{n-1} (-1)^{i-1} e_i h_{n-i}. \tag{9}$$

The right hand side of (9) is equal to

$$\begin{aligned}(-1)^{n-1} e_n + \sum_{i=1}^{n-1} (-1)^{i-1} e_i \sum_{\alpha \vdash n-i} M_{\alpha,(n-i)} e_\alpha \\ = (-1)^{n-1} e_n + \sum_{\lambda \vdash n} \left(\sum_{i=1}^{n-1} (-1)^{i-1} M_{\lambda\setminus i,(n-i)} \right) e_\lambda.\end{aligned}$$

Picking the coefficient of e_n on the right hand side of the above equation, $M_{(n),(n)} = (-1)^{n-1}$. Moreover, $M_{\lambda,(n)} = \sum_{i=1}^{n-1} (-1)^{i-1} M_{\lambda\setminus i,(n-i)}$. This verifies items 1 and 2 on our list. As for item 3, consider

$$\begin{aligned}\sum_{\lambda \vdash n} M_{\lambda,\mu} e_\lambda &= h_{(\mu_1)} h_{\mu\setminus\mu_1} \\ &= \sum_{\alpha \vdash \mu_1} M_{\alpha,(\mu_1)} e_\alpha \sum_{\beta \vdash n-\mu_1} M_{\beta,\mu\setminus\mu_1} e_\beta \\ &= \sum_{\substack{\alpha \vdash \mu_1 \\ \beta \vdash n-\mu_1}} M_{\alpha,(\mu_1)} M_{\beta,\mu\setminus\mu_1} e_\alpha e_\beta. \end{aligned} \tag{10}$$

Comparing the coefficient on both sides of (10) shows item 3.

The list items 1–3 completely determine the numbers $M_{\lambda,\mu}$ recursively. To complete the proof of the theorem, it remains to be shown that $(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}|$ satisfy the same three identities.

There is only one brick tabloid of shape (n) and type (n)—the brick tabloid consisting of one brick of length n inside one row of length n. Therefore, when $\lambda, \mu = (n)$, $(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}| = (-1)^{n-1}$, verifying item 1.

Item 2 is found by sorting brick tabloids of shape (n) according to the length of the first brick. Suppose $\lambda \neq (n)$ and i is a part of λ. Let $B_{\lambda,(n),i}$ be the set of $T \in B_{\lambda,(n)}$ where the first brick in T has length i. It follows that $|B_{\lambda,(n),i}| = |B_{\lambda\setminus i,(n-i)}|$. Thus,

$$\begin{aligned}(-1)^{n-\ell(\lambda)}|B_{\lambda,(n)}| &= (-1)^{n-\ell(\lambda)}\sum_{i=1}^{n-1}|B_{\lambda,(n),i}| \\ &= \sum_{i=1}^{n-1}(-1)^{i-1}\left((-1)^{(n-i)-(\ell(\lambda)-1)}|B_{\lambda\setminus i,(n-i)}|\right),\end{aligned}$$

verifying item 2.

Finally, item 3 is found by sorting brick tabloids of shape μ according to the bricks found in the top row. Suppose $B_{\lambda,\mu,\alpha}$ is the set of all $T \in B_{\lambda,\mu}$ where the first row in T has bricks which induce the partition α. It follows that $|B_{\lambda,\mu,\alpha}| = |B_{\alpha,(\mu_1)}||B_{\beta,\mu\setminus\mu_1}|$ where $\beta = \lambda - \alpha$ and therefore

$$(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}| = \sum_{\substack{\alpha\vdash\mu_1,\ \beta\vdash n-\mu_1 \\ \alpha+\beta=\lambda}}(-1)^{\mu_1-\ell(\alpha)}|B_{\alpha,(\mu_1)}|(-1)^{(n-\mu_1)-\ell(\beta)}|B_{\beta,\mu\setminus\mu_1}|.$$

This checks item 3 and completes the proof of the theorem. $\square$

One can find the transition matrix $M(h,e)$ is a similar fashion. Indeed, by the symmetry of the relation between the h's and e's in Lemma 1, it is easy to see that $M(h,e) = M(e,h)$.

2.2 Previous Results

We list here some results from the literature which follow from the same simple identity as the distance 2 results we will present. We state first some identities that are needed to understand the results.
Let $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, then

$$\begin{array}{ll} Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} & Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\} \\ des(\sigma) = |Des(\sigma)| & rise(\sigma) = |Rise(\sigma)| \\ maj(\sigma) = \sum_{i\in Des(\sigma)} i & comaj(\sigma) = \sum_{i\in Rise(\sigma)} i \\ rlmaj(\sigma) = \sum_{i\in Des(\sigma)} n-i & rlcomaj(\sigma) = \sum_{i\in Rise(\sigma)} n-i \\ inv(\sigma) = \sum_{i<j}\chi(\sigma_i > \sigma_j) & coinv(\sigma) = \sum_{i<j}\chi(\sigma_i < \sigma_j) \\ exc(\sigma) = |\{i : i < \sigma_i\}| & dec(\sigma) = |\{i : i > \sigma_i\}| \end{array}$$

where for any statement A, $\chi(A)=1$ if A is true and $\chi(A)=0$ if A is false. Also if $\alpha^1,\ldots,\alpha^k \in S_n$, then we shall write $comdes(\alpha^1,\ldots,\alpha^k) = |\bigcap_{i=1}^k Des(\alpha^i)|$.
Some identities needed for the q-analogues:
$[n]_q = 1+q+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$

$[n]_q! = [n]_q[n-1]_q\cdots[1]_q$

$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q![n-k]_q!}$

$\begin{bmatrix} n \\ \lambda_1,\ldots,\lambda_\ell \end{bmatrix} = \frac{[n]_q!}{[\lambda_1]_q!\cdots[\lambda_\ell]_q!}$

$[n]_{p,q} = p^{n-1}+p^{n-2}q+\cdots+p^1q^{n-2}+q^{n-1} = \frac{p^n-q^n}{p-q}$.

Using these facts the following results have been shown:

1. $\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma\in S_n} x^{des(\sigma)} = \frac{1-x}{-x+e^{u(x-1)}}$

2. (Carlitz 1970) $\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau)\in S_n\times S_n} x^{comdes(\sigma,\tau)} = \frac{1-x}{-x+J(u(x-1))}$.

3. (Stanley 1979) $\sum_{n=0}^{\infty} \frac{u^n}{[n]!} \sum_{\sigma\in S_n} x^{des(\sigma)}q^{inv(\sigma)} = \frac{1-x}{-x+e_q(u(x-1))}$.

4. (Stanley 1979) $\sum_{n=0}^{\infty} \frac{u^n}{[n]!} \sum_{\sigma\in S_n} x^{des(\sigma)}q^{coinv(\sigma)} = \frac{1-x}{-x+E_q(u(x-1))}$.

5. (Fedou and Rawlings 1995)
   $\sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau)\in S_n\times S_n} x^{comdes(\sigma,\tau)}q^{inv(\sigma)}p^{inv(\tau)} = \frac{1-x}{-x+J_{q,p}(u(x-1))}$.

   $J(u) = \sum_{n\geq 0} \frac{u^n}{n!n!}, \qquad e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} q^{\binom{n}{2}},$

   $E_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}$, and $\qquad J_{q,p}(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} q^{\binom{n}{2}} p^{\binom{n}{2}}$.

6. Foata-Han (1997)
   Let $(x,q)_0 = 1$ and $(x,q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$ for $n>0$.

$$C_n(z,x,q,y,p) = \sum_{(\sigma,\tau)\in S_n\times S_n} z^{comdes(\sigma^{-1},\tau^{-1})}x^{des(\sigma)}q^{rlmaj(\sigma)}y^{rise(\tau)}p^{rlcomaj(\tau)}.$$

$$\sum_{n\geq 0} t^n \frac{C_n(z,x,q,y,p)}{(x,q)_{n+1}(y,p)_{n+1}} = \sum_{i,j\geq 0} \frac{x^i y^j}{1+\sum_{n\geq 1}(t(z-1))^{n-1} \begin{bmatrix} i+1 \\ n \end{bmatrix}_q \begin{bmatrix} j+n \\ n \end{bmatrix}_p}.$$

7. Remmel-Mendes
   $R_n(z,x,q,y,p,Q,P) =$
   $\sum_{(\alpha,\beta,\gamma,\delta)\in S_n^4} z^{comdes(\alpha^{-1},\beta^{-1},\gamma,\delta)} x^{des(\alpha)} q^{rlmaj(\alpha)} y^{rise(\beta)} p^{rlcomaj(\beta)} Q^{inv(\gamma)} P^{coinv(\delta)}$
   and set

$$F^{i,j}(t,q,p,Q,P) = \sum_{n\geq 0} t^n \frac{q^{\binom{n}{2}} Q^{\binom{n}{2}} \begin{bmatrix} i+1 \\ n \end{bmatrix}_q \begin{bmatrix} j+n \\ n \end{bmatrix}_p}{[n]_Q![n]_P!}.$$

We end this section with an example borrowed from Mendes and Remmel [11] which show how we can apply a ring homormorphism ξ^{f_1} from Λ into the polynomial ring $Q[x]$ over the rationals Q to a simple symmetric function identity to produce generating functions for permutation statistics. Since $e_0, e_1, \dots$ are algebraically independent and generate Λ, we can define our homorphism ξ^{f_1} by simply giving the values of $\xi^{f_1}(e_n)$ for each n.

Let f_1 be a function on the nonnegative integers such that $f_1(n) = 1$ if $n = 0$ and $f_1(n) = -y(x-y)^{n-1}$ if $n \geq 1$ and define a ring homomorphism $\xi^{f_1} : \Lambda_n \to \mathbb{Q}[x,y]$ such that for $n \geq 1$,

$$\xi^{f_1}(e_n) = \frac{(-1)^n}{n!} f_1(n) \tag{11}$$

This definition uniquely extends to all of Λ because products of elementary symmetric functions are a basis. This homomorphism and its relationship to Theorem 4 below are due to Brenti; however, the proof hinges on ideas established by Beck and Remmel when they reproved the results of Brenti combinatorially [1, 3, 4, 5]. Our entire development of finding generating functions will come from these ideas.

Given $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let the rise statistic, $\mathrm{ris}(\sigma)$, count the number of times $\sigma_i < \sigma_{i+1}$. By convention, let $\sigma_{n+1} = n+1$ so that σ_n always registers a rise.

Theorem 4. †

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} = \frac{x-y}{x-y\mathbf{e}^{t(x-y)}}.$$

†A boldface $\mathbf{e}$ is used to distinguish the exponential function from the elementary symmetric function.

Proof. First it will be shown that

$$n!\xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)}$$

after which the statement of the theorem follows shortly. To evaluate ξ^{f_1} on $n!h_n$, write h_n in terms of the elementary symmetric functions via Theorem 3:

$$\begin{aligned}
n!\xi^{f_1}(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \xi^{f_1}(e_{\lambda_i}) \\
&= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i}}{\lambda_i!} f_1(\lambda_i) \\
&= \sum_{\lambda \vdash n} \binom{n}{\lambda} |B_{\lambda,(n)}| y^{\ell(\lambda)} (x-y)^{n-\ell(\lambda)} \qquad (12)
\end{aligned}$$

where if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$\binom{n}{\lambda} = \binom{n}{\lambda_1, \ldots, \lambda_\ell} = \frac{n!}{\lambda_1! \cdots \lambda_\ell!}$$

is the usual multinomial coefficient.

(12) will be interpreted as a signed, weighted sum of objects on which a sign-reversing, weight-preserving involution will be performed. The fixed points under the involution will correspond to elements in S_n with the weights on the fixed point giving the number of descents and rises in the permutation.

The sum in (12) selects $\lambda \vdash n$. Use the $|B_{\lambda,(n)}|$ term in (12) to select a brick tabloid of shape (n) filled with bricks forming the partition λ. With the multinomial coefficient, select λ_1 integers from $1, \ldots, n$ to place in a brick of length λ_1 in decreasing order, λ_2 of the remaining integers to place in a brick of length λ_2 in decreasing order, etc., so that each brick contains a decreasing sequence and each integer in $1, \ldots, n$ appears once. The $(x-y)^{n-\ell(\lambda)}$ term in the sum in (12) is used to label each cell not terminating a brick with either x or $-y$. Finally, place a y in each terminal cell in a brick. The set of all such objects able to be formed in this way will be denoted $\mathcal{T}_{\xi^{f_1}}$. An example of one such $T \in \mathcal{T}_{\xi^{f_1}}$ may be found below.

x	x	$-y$	y	$-y$	y	x	y	y	$-y$	x	y
12	10	8	2	7	1	6	5	3	11	9	4

Define the weight of $T \in \mathcal{T}_{\xi^{f_1}}$, $w(T)$, to be the product of the $x, -y$, and y labels in T. The above example has weight $(-1)^3 x^4 y^8$. We have accounted for every term in (12); therefore,

$$n!\xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{\xi^{f_1}}} w(T).$$

At this point, a sign-reversing, weight-preserving involution $I_{\xi^{f_1}}$ will be defined on $\mathcal{T}_{\xi^{f_1}}$ to leave a set of fixed points with positive sign. Let $T \in \mathcal{T}_{\xi^{f_1}}$. Scan T from left to right looking for the first of the following two occurrences:

1. a cell labeled with $-y$, or

2. two consecutive bricks with a decrease in the labeling between them.

If situation 1 appears first, break the brick containing the $-y$ into two bricks immediately after the violation and change the $-y$ to a y. If situation 2 appears first, combine the two consecutive bricks and change the y now in the middle of the brick to a $-y$. This process is the involution $I_{\xi^{f_1}}$—it does not alter any cells labeled with x but does flip the sign on T. The image of the object found earlier in this proof under $I_{\xi^{f_1}}$ is displayed below.

x	x	y	y	$-y$	y	x	y	y	$-y$	x	y
12	10	8	2	7	1	6	5	3	11	9	4

Let $\mathcal{F}_{\xi^{f_1}}$ be the set of fixed points under the involution $I_{\xi^{f_1}}$ consisting of those $T \in \mathcal{T}_{\xi^{f_1}}$ where there are no $-y$'s and there are no decreases between two bricks. An example of $T \in \mathcal{F}_{\xi^{f_1}}$ may be found below.

x	x	x	y	x	y	x	y	y	x	x	y
12	10	8	2	7	1	6	3	5	11	9	4

The row of integers on a fixed point can be read as an element of the symmetric group S_n written in one line notation. When this is done, there is an x label above an integer if and only if that integer registers a descent and a y label above an integer if and only if that integer registers a rise. The above fixed point corresponds to 12 10 8 2 7 1 6 3 5 11 9 4 $\in S_{12}$ with seven descents and five rises. The involution $I_{\xi^{f_1}}$ implies

$$n!\xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{\xi^{f_1}}} w(T) = \sum_{T \in \mathcal{F}_{\xi^{f_1}}} w(T) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)}.$$

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} &= \sum_{n=0}^{\infty} t^n \xi^{f_1}(h_n) \\
&= \xi^{f_1}\left(\sum_{n=0}^{\infty} h_n t^n\right) \\
&= \xi^{f_1}\left(\sum_{n=0}^{\infty} e_n (-t)^n\right)^{-1}
\end{aligned}$$

where the last equality comes from (5). Continuing this string of equalities, we have

$$\frac{1}{1+\sum_{n=1}^{\infty}(-t)^n(-1)^n\frac{-y(x-y)^{n-1}}{n!}} = \frac{x-y}{x-y-y\sum_{n=1}^{\infty}\frac{t^n(x-y)^n}{n!}} = \frac{x-y}{x-y\mathbf{e}^{t(x-y)}}.$$

□

3 Statistics on pairs

This section covers the new results obtained during the course of the project.

3.1 $des^{(2)}(\sigma)$

In this section we wish to consider permutation statistics on pairs of elements. More specifically, if $\sigma \in S_n$ and $\sigma = \sigma_1\sigma_2...\sigma_n$ then we want to look at the relation between $\{\sigma_{i-1}, \sigma_i\}$ and $\{\sigma_{i+1}, \sigma_{i+2}\}$. To do this we must first define the set of $\sigma \in S_n$ that we will work on.

Definition 1.

$$\mathcal{E}_{2n} = \{\sigma \in S_{2n} : \textit{maximal descent blocks are even}\}$$

This gives us a set of σ that are of even length and can be split into blocks of even length.
Next we must define a new permutation statistic on $\mathcal{E}_{2n}$ by comparing descents on pairs of σ_i for $\sigma \in \mathcal{E}_{2n}$ in the following way:

Definition 2. $Des^{(2)}(\sigma) = \{i : \sigma_i > \sigma_{i+2}\, and\, \sigma_{i-1} > \sigma_{i+1}\}$

As an example consider $\tau = 8\,6\,4\,3\,2\,9\,10\,7\,5\,1$. Then $Des^{(2)}(\tau) = \{2, 8\}$. As before we primarily wish to consider the cardinality of $Des^{(2)}(\sigma)$.

Definition 3. $des^{(2)}(\sigma) = |Des^{(2)}(\sigma)|$

Using the same τ from before we have $des^{(2)}(\tau) = 2$.
Finally we introduce a new homomorphism, $\varphi^{(2)} : \Lambda_n \to \mathbb{Q}[x]$ such that

$$\begin{aligned}
\varphi^{(2)}(e_0) &= 1 \\
\varphi^{(2)}(e_{2n}) &= \frac{(-1)^n(1-x)^{n-1}}{2n!}\binom{2n}{n} = \frac{-(x-1)^{n-1}}{n!n!} \\
\varphi^{(2)}(e_{2n+1}) &= 0
\end{aligned}$$

We use these definitions and homomorphism to show the following:

Theorem 5.

$$(2n)!\,\varphi^{(2)}(h_{2n}) = \sum_{\sigma \in \mathcal{E}_{2n}} x^{des^{(2)}(\sigma)}$$

Proof. First, by Theorem 3 we can write

$$n!\varphi^{(2)}(h_n) = n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}|B_{\mu,n}|\varphi^{(2)}(e_\mu). \tag{13}$$

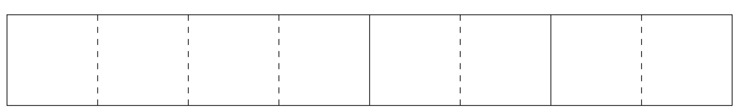
Since $\varphi^{(2)}(e_{2n+1}) = 0$, it is easy to see that if μ has an odd part, then $\varphi^{(2)}(e_\mu) = 0$. It follows that $\varphi^{(2)}(h_{2n+1}) = 0$ for all $n \geq 0$. Similarly, the only μ that can contribute the sum in (13) when n is even are the partitions μ which consist entirely of even parts. If $\mu = (\mu_1, \ldots, \mu_k)$ is a partition of n, then we write 2μ for the partition $(2\mu_1, \ldots, 2\mu_k)$. It is then easy to see that

$$(2n)!\,\varphi^{(2)}(h_{2n}) = (2n)!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)}|B_{2\mu,(2n)}|\varphi^{(2)}(e_{2\mu}).$$

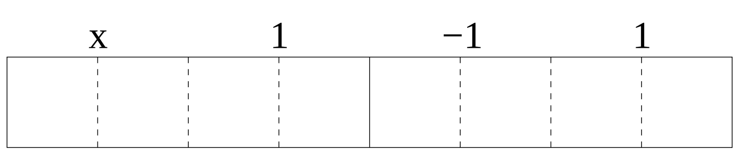
Using the definition of $\varphi^{(2)}$ we then have

$$\begin{aligned}(2n)!\,\varphi^{(2)}(h_{2n}) &= (2n)!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)}|B_{2\mu,(2n)}|\prod_{i=1}^{\ell(\mu)}\frac{-(x-1)^{\mu_i-1}}{2\mu_i!}\binom{2\mu_i}{\mu_i}\\ &= (2n)!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)}|B_{2\mu,(2n)}|(-1)^{\ell(\mu)}(x-1)^{n-\ell(\mu)}\frac{1}{2\mu_1!2\mu_2!...2\mu_{\ell(\mu)}!}\prod_{i=1}^{\ell(\mu)}\binom{2\mu_i}{\mu_i}\\ &= \sum_{\mu\vdash n}|B_{2\mu,(2n)}|\binom{2n}{2\mu_1 2\mu_2 ... 2\mu_{\ell(\mu)}}(x-1)^{n-\ell(\mu)}\prod_{i=1}^{\ell(\mu)}\binom{2\mu_i}{\mu_i} \qquad (14)\end{aligned}$$

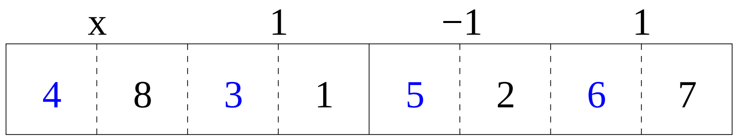
We can now interpret (14) above as a combinatorial object starting with $B_{2\mu,(2n)}$. We use this to construct bricks of length $2n$ and shape μ with each partition inside the brick of size $2\mu_i$ where $\mu = (\mu_1\,\mu_2\,...\mu_{\ell(\mu)})$. An example of such a brick is given below for $n = 4$ and $\mu = (2, 1, 1)$.



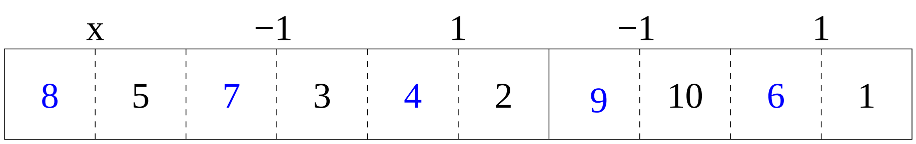
Next, we interpret $(x-1)^{n-\ell(\mu)}$ by putting a -1 or an x over each pair of parts of a brick up to the last pair of the brick. We fill a 1 at the end of each brick. Our brick from above now looks like this:



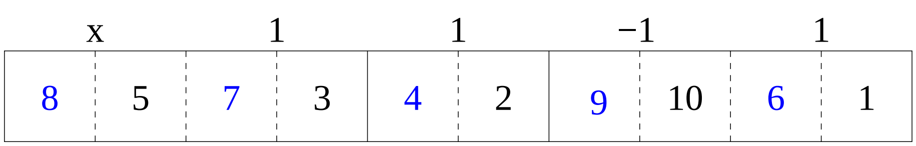
We interpret $\binom{2n}{2\mu_1 2\mu_2 \ldots 2\mu_{\ell(\mu)}}$ by choosing sets of length $2\mu_i$ from $[2n] = \{1, ..., 2n\}$. Continuing the example from above our sets might look something like $\{1,3,4,8\}$, $\{2,5\}$, $\{6,7\}$. For each i we intepret $\binom{2\mu_i}{\mu_i}$ by choosing half of the numbers out of each set we have. We color these with blue to denote they have been chosen. So now we have the sets $\{1,3,4,8\}$, $\{2,5\}$, $\{6,7\}$. Starting with the blue numbers we fill in every other open portion of the brick in descending order. We then fill in the regular numbers in descending order in the remaining spaces. Our object now looks like this:



Call this object T and let $\mathcal{T}_\varphi$ be the set of all such objects. The weight of T, $w(T)$, is the product of the row of x's and 1's above the brick. We can also read off the numbers inside the brick from left to right to form a permutation $\sigma = 4\,8\,3\,1\,5\,2\,6\,7$. We are interested in $\sum_T w(T)$ so we will use an involution I to get rid of the T with negative weight. The involution I works similar to the one used before. First we scan from left to right looking for either a -1 or a pair descent between bricks. If we find a -1 then split the brick at the rightmost element of the pair under the -1 and change the -1 to 1. If we encounter a pair descent first then join the two bricks and change the trailing 1 of the first brick to a -1. As an example we use the following object T:



Applying I gives us T':

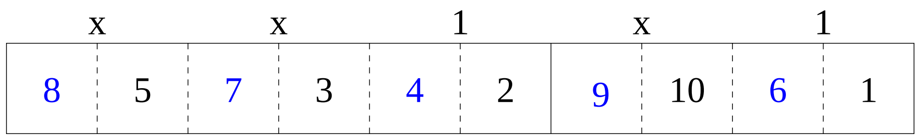


Applying I to T' gives us back T since there is a pair descent between $7\,3$ and $4\,2$ which means that I will join the bricks and change the sign of the 1.

The weight of T is $x \cdot -1 \cdot 1 \cdot -1 \cdot 1 = x$. The weight of T' is $x \cdot 1 \cdot 1 \cdot -1 \cdot 1 = -x$. So our involution I is sign reversing and for T not a fixed point we have $w(T) = -w(I(T))$. From this we can see that

$$\sum_{T \in \mathcal{T}_\varphi} w(T) = \sum_{T \text{ a fixed point of } I \in \mathcal{T}_\varphi} w(T)$$

A fixed point of I must have no -1's and no pair descents between bricks. An example of a fixed point is given here:



Fixed points then have an x everywhere except the last pair of a brick and have no pair descents between bricks. If we read off the numbers of the fixed point from left to right we get a permutation $\sigma = 8\,5\,7\,3\,4\,2\,9\,10\,6\,1$. We can observe then that on our object the x's correspond to the pairs where a pair descent is registered and the 1's to a pair where no pair descent exists. It follows then that

$$\sum_{T \text{ a fixed point of } I \in \mathcal{T}_\varphi} w(T) = (2n)!\,\varphi^{(2)}(h_{2n})$$

and from above,

$$\sum_{T \text{ a fixed point of } I} w(T) = \sum_{\sigma \in \mathcal{E}_{2n}} x^{des^{(2)}(\sigma)}$$

So we have shown that Theorem 5 is true. $\square$

Next we use this result to show a new generating function. Start by letting $J(u) = \sum_{n \geq 0} \frac{u^n}{n!n!}$. Because $\varphi^{(2)}(h_{2n+1}) = 0$ for all n, it follows that

$$\begin{aligned} \varphi^{(2)}\left(\sum_{n\geq 0} h_n t^n\right) &= \varphi^{(2)}\left(\sum_{n \geq 0} h_{2n} t^{2n}\right) \\ &= \sum_{n\geq 0} \frac{t^{2n}}{2n!} \sum_{\sigma \in \mathcal{E}_{2n}} x^{des^{(2)}(\sigma)}. \end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n\geq 0}\frac{t^{2n}}{2n!}\sum_{\sigma\in\mathcal{E}_{2n}} x^{des^{(2)}(\sigma)} &= \varphi^{(2)}\left(\sum_{n\geq 0} h_n t^n\right)\\
&= \varphi^{(2)}\left(\frac{1}{\sum_{n\geq 0} e_n(-t)^n}\right)\\
&= \frac{1}{1+\sum_{n\geq 1}\frac{-(x-1)^{n-1}}{n!n!}(t)^{2n}}\\
&= \frac{-1}{-1+\sum_{n\geq 1}\frac{(x-1)^{n-1}}{n!n!}(t)^{2n}}\\
&= \frac{-(x-1)}{-(x-1)+\sum_{n\geq 1}\frac{(x-1)^{n}}{n!n!}(t)^{2n}}\\
&= \frac{-(x-1)}{-(x-1)+(J((x-1)t^2)-1)}\\
&= \frac{(1-x)}{-x+J((x-1)t^2)}
\end{aligned}$$

3.2 $\overline{des^{(2)}}(\sigma)$

In the previous section we considered the case where there was a descent registered between two pairs of elements in some $\sigma \in \mathcal{E}_{2n}$. We can also however consider what happens when we count descents between only the elements that have an odd or even position. To do this we introduce first a new set of permutations that we will use.

Definition 4.

$$\mathcal{F}_{2n} = \{\sigma \in S_{2n} : \textit{maximal descent blocks on even elements are even}\}$$

Then as before we will define a new statistic on this set $\mathcal{F}_{2n}$.

Definition 5. $\overline{Des^{(2)}}(\sigma) = \{i : i \textit{ odd and } \sigma_i > \sigma_{i+2}\}$

As an example let $\sigma = 10\,5\,7\,2, 8\,9\,4\,3\,1\,6$. Then $\overline{Des^{(2)}}(\sigma) = \{1, 5, 7\}$. Note that we can easily modify this definition to count descents between σ_i where i is even. Again we will concern ourselves with the cardinality of this set.

Definition 6. $\overline{des^{(2)}}(\sigma) = |\overline{Des^{(2)}}(\sigma)|$

The homomorphism we use is very similar to the one before however we now must account for all the different ways in which we can order the set of numbers, in this case those in even places, that we are not concerned with statistics on. We call this new homomorphism $\Psi^{(2)}$ and define $\Psi^{(2)} : \Lambda_n \to \mathbb{Q}[x]$ in the following way:

$$\begin{aligned}
\Psi^{(2)}(e_0) &= & 1 & \\
\Psi^{(2)}(e_{2n}) &= & \tfrac{(-1)^n(1-x)^{n-1}}{2n!}\tbinom{2n}{n}n! = & \frac{-(x-1)^{n-1}}{n!} \\
\Psi^{(2)}(e_{2n+1}) &= & 0 &
\end{aligned}$$

We now put all of this together by again stating a new theorem and proving it.

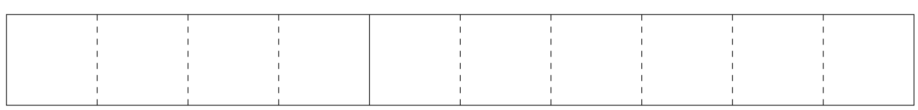
Theorem 6.

$$(2n)!\,\Psi^{(2)}(h_{2n}) = \sum_{\sigma \in \mathcal{F}_{2n}} x^{\overline{des^{(2)}}(\sigma)}$$

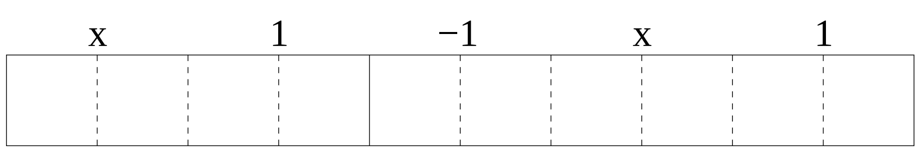
Proof. Using the same argument as in Theorem 5 we can write:

$$\begin{aligned}
(2n)!\,\Psi^{(2)}(h_{2n}) &= 2n! \sum_{\mu \vdash n} (-1)^{(2n-\ell(\mu))} |B_{2\mu,(2n)}| \Psi^{(2)}(e_{2n}) \\
&= 2n! \sum_{\mu \vdash n} (-1)^{\ell(\mu)} |B_{2\mu,(2n)}| \prod_{i=1}^{\ell(\mu)} \frac{(-1)(x-1)^{\mu_i-1}}{2\mu_i!} \binom{2\mu_i}{\mu_i} \mu_i! \\
&= \sum_{\mu \vdash n} |B_{2\mu,(2n)}| (x-1)^{n-\ell(\mu)} \binom{2n}{2\mu_i \ldots 2\mu_{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} \binom{2\mu_i}{\mu_i} \mu_i!
\end{aligned}$$

We now will construct a set of objects $\mathcal{T}_\Psi$. First we build a set of bricks of length $2n$ and shape μ with each section being of size $2\mu_i$. If we have $\mu = (2,3)$ then the corresponding brick is:



We now interpret $(x-1)^{n-\ell(\mu)}$ as before; by placing either an x or a -1 above each pair of boxes in the brick up to the last pair in the section. An example of placements is given below:



Next we choose sets of length $2\mu_i$ from $[2n]$. Using $\mu = (2,3)$ our sets might be $\{5,3,8,10\}$, $\{1,2,7,4,6,9\}$. In each set we pick a new sequence of length μ_i and color them blue. The sets we picked now look like this: $\{5,3,8,10\}$, $\{1,2,7,4,6,9\}$. We can fill in the bricks now starting with the blue numbers in descending order in the odd places. After this is done we fill in the black numbers in any order, with the term $\mu_i!$ accounting for the possible ways to fill in the black numbers in each section of the brick. A filled brick looks like this:

x		1		x		x		1	
5	10	3	8	6	1	4	9	2	7

We will call this object T and let $\mathcal{T}_\Psi$ be the set of all such objects constructed in this manner.
As before we define the weight of an object T, $w(T)$ as the product of the x's and -1's above it. We then use the same involution I to get rid of all the negative weights in the sum $\sum_{T\in\mathcal{T}_\Psi} w(T)$. As before I works in the following manner:

1. Scan the bricks from left to right for a -1 or a descent between two odds in consecutive bricks
2. If a -1 is encountered, break the brick at the second box below the -1 and change the -1 to a 1.
3. If a descent between two odds in consecutive bricks is found, join the bricks and change the trailing 1 of the first brick to a -1.

As before this process is reversible, sign changing and weight preserving. After applying I we see that $\sum_{T\in\mathcal{T}_\Psi} w(T) = \sum_{T \text{ a fixed point in } \mathcal{T}_\Psi} w(T)$. In this case fixed points are those objects with an x above every pair of boxes except the last in a brick and no descents between odd places of consecutive bricks. An example of such an object is this:

x		1		x		x		1	
5	10	3	8	6	1	4	9	2	7

We have now shown that

$$(2n)!\,\Psi^{(2)}(h_{2n}) = \sum_{T \text{ a fixed point in } \mathcal{T}_\Psi} w(T)$$

If we then look at a fixed point in $\mathcal{T}_\Psi$ such as the one above we can also read off a permutation σ. In this case $\sigma = 5\,10\,3\,8\,6\,1\,4\,9\,2\,7$. We can see that $\overline{Des^{(2)}}(\sigma) = \{1,5,7\}$ which correspond to the odd boxes with an x above them in our object T so we have:

$$\sum_{T \text{ a fixed point in } \mathcal{T}_\Psi} w(T) = \sum_{\sigma \in \mathcal{F}_{2n}} x^{\overline{des^{(2)}}(\sigma)}$$

Which then implies

$$(2n)!\,\Psi^{(2)}(h_{2n}) = \sum_{\sigma \in \mathcal{F}_{2n}} x^{\overline{des^{(2)}}(\sigma)}$$

So we have proved Theorem 6. $\square$

Again we use this result to show a new generating function. Since $\varphi^{(2)}(h_{2n+1}) = 0$ for all n, it follows that

$$\begin{aligned}
\Psi^{(2)}\left(\sum_{n\geq 0} h_n t^n\right) &= \Psi^{(2)}\left(\sum_{n\geq 0} h_{2n} t^{2n}\right) \\
&= \sum_{n\geq 0} \frac{t^{2n}}{2n!} \sum_{\sigma \in \mathcal{F}_{2n}} x^{\overline{des^{(2)}}(\sigma)}.
\end{aligned}$$

And so we can write:

$$\begin{aligned}
\sum_{n\geq 0} \frac{t^{2n}}{2n!} \sum_{\sigma \in \mathcal{F}_{2n}} x^{\overline{des^{(2)}}(\sigma)} &= \Psi^{(2)}\left(\sum_{n\geq 0} h_n t^n\right) \\
&= \Psi^{(2)}\left(\frac{1}{1+\sum_{n\geq 1} e_n(-t)^n}\right) \\
&= \frac{1}{1+\sum_{n\geq 1} \frac{-(x-1)^{(n-1)}}{n!}(-t)^{2n}} \\
&= \frac{-1}{-1+\sum_{n\geq 1} \frac{(x-1)^{(n-1)}}{n!}(t)^{2n}} \\
&= \frac{-(x-1)}{-(x-)1+\sum_{n\geq 1} \frac{(x-1)^{n}}{n!}(t)^{2n}} \\
&= \frac{1-x}{-x+\mathbf{e}^{t(x-y)^{(x-1)t^2}}}
\end{aligned}$$

3.3 q-analogue of $des^{(2)}(\sigma)$

In this section we extend the previous result using $des^{(2)}(\sigma)$ to one with q-analogues. To do this we need to define a new statistic that will count inversions on every other object.

Definition 7.

$$inv^{(2)}(\sigma) = \sum_{0 \leq i < j \leq n} \chi(\sigma_{2i} > \sigma_{2j}) + \chi(\sigma_{2i+1} > \sigma_{2j+1})$$

As an example consider $\sigma = 10\,5\,7\,2, 8\,9\,4\,3\,1\,6$. We see then that $inv^{(2)}(\sigma) = 4 + 2 + 2 + 1 = 9$.

We also will make use of the following identities:

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

$$\begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_\ell \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!}$$

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + p^1 q^{n-2} + q^{n-1} = \frac{p^n - q^n}{p-q}$$

Using these identities we now define a new homormorphism $\varphi_q^{(2)}\Lambda_n \to \mathbb{Q}[x,q]$:

$$\begin{aligned} \varphi_q^{(2)}(e_0) &= 1 \\ \varphi_q^{(2)}(e_{2n+1} &= 0 \\ \varphi_q^{(2)}(e_{2n}) &= \frac{(-1)^n(1-x)^{n-1}}{[2n]_q}\begin{bmatrix} 2n \\ n \end{bmatrix} q^{2\binom{n}{2}} = \frac{-(x-1)^{n-1}}{[n]_q![n]_q!} q^{2\binom{n}{2}} \end{aligned}$$

Using this homomorphism we will prove the following theorem:

Theorem 7.

$$\binom{2n}{n}[2n]_q!^2 \varphi_q^{(2)}(h_{2n}) = \sum_{\sigma \in \mathcal{E}_{2n}} x^{des^{(2)}(\sigma)} q^{inv^{(2)}(\sigma)}$$

Proof. Let $\mathcal{R}(1^{\mu_1}, \cdots, \ell^{\mu_\ell})$ be the set of rearrangements of μ_1 1's, μ_2 2's, ... Then a theorem due to Carlitz [6] gives us:

$$\begin{bmatrix} n \\ \mu_1, \cdots, \mu_\ell \end{bmatrix}_q = \sum_{r \in \mathcal{R}(1^{\mu_1}, \cdots, \ell^{\mu_\ell})} q^{inv(r)}$$

From this we arrive at the following result:

Theorem 8. *Let T be a brick tabloid of shape μ and size n. Then S_n, T is the set of all σ in S_n that are the result of placing decreasing sequences of integers in the brick of T, and*

$$\sum_{\sigma \in S_n, T} q^{inv(\sigma)} = \begin{bmatrix} n \\ \mu_1 \cdots \mu_\ell \end{bmatrix}_q q^{\sum_i \binom{\mu_i}{2}}$$

To show this we start by taking an $r \in \mathcal{R}(1^{\mu_1}, \cdots, \ell^{\mu_\ell})$. As an example let $\mu = (2,3,3)$ and $r = 2\,1\,2\,2\,2\,3\,1\,3\,3$. We can see that $inv(r) = 5$ in this example. Now we proceed by filling in the 1's of r from right to left with numbers from 1 to n, in our example 8, until we have no 1's left. We then do the same for the 2's, filling them in right to left with numbers from μ_1 to n, etc. When we have filled in all of the numbers we call this $\sigma(r)$.

$$\begin{aligned} r &= 2\,1\,2\,2\,3\,1\,3\,3 \\ \sigma(r) &= 5\,2\,4\,3\,8\,1\,7\,6 \\ \sigma(r)^{-1} &= 6\,2\,4\,3\,1\,8\,7\,5 \end{aligned}$$

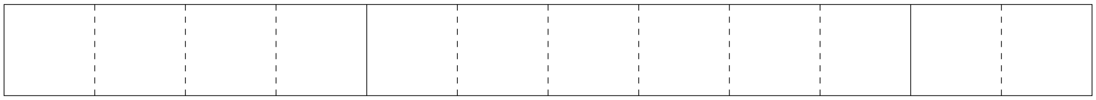
We see then that every time an inversion was registered in r it is registered in $\sigma(r)$. We have new inversions as well. Due to the way we filled in the numbers, within each block of μ_i's in r we also generate $(\mu_i - 1) + (\mu_i - 2) + \ldots + (\mu_i - (\mu_i - 2) + 1 = \binom{\mu_i}{2}$ new inversions. Putting this together we have $inv(\sigma(r)) = inv(r) + \sum_i \binom{\mu_i}{2}$. We also know that $inv(\sigma) = inv(\sigma)^{-1}$ and since $\sigma(r)^{-1}$ is in decreasing blocks of length μ_i we can write:

$$\begin{aligned} \begin{bmatrix} n \\ \mu_1, \cdots, \mu_\ell \end{bmatrix}_q q^{\sum_i \binom{\mu_i}{2}} &= q^{\sum_i \binom{\mu_i}{2}} \sum_{r \in \mathcal{R}(1^{\mu_1}, \cdots, \ell^{\mu_\ell})} q^{inv(r)} \\ &= \sum_{\sigma \in S_n, T} q^{inv(\sigma)} \end{aligned}$$

which proves Theorem 8.
We proceed now by following the same method used for the proof of Theorem 5. First, we use Theorem 3 to set the left hand side equal to something we can interpret:

$$\begin{aligned} \binom{2n}{n}[2n]_q!^2 \varphi_q^{(2)}(h_{2n}) &= \binom{2n}{n}[2n]_q!^2 \sum_{\mu \vdash n} (-1)^{2n-\ell(\mu)} B_{\mu,(2n)} \varphi_q^{(2)}(e_{2n}) \\ &= \binom{2n}{n}[n]_q!^2 \sum_{\mu \vdash n} (-1)^{2n-\ell(\mu)} B_{\mu,(2n)} \prod_{i=1}^{\ell(\mu)} \frac{-(x-1)^{\mu_i - 1}}{[\mu_i]_q![\mu_i]_q!} q^{2\binom{\mu_i}{2}} \\ &= \sum_{\mu \vdash n} B_{\mu,(2n)} (x-1)^{n-1} \binom{2n}{n} \begin{bmatrix} n \\ \mu_1, \cdots, \mu_{\ell(\mu)} \end{bmatrix}_q^2 q^{2\sum_{i=1}^{\ell(\mu)} \binom{\mu_i}{2}} \end{aligned}$$

To interpret the RHS we start with $\mu \vdash n$. For an example let us use $\mu = (2,3,1)$. From this we then set out a brick of shape $2\mu = (4,6,2)$ and size $2n = 12$.



This takes care of the term $B_{2\mu,(2n)}$. Next we use the term $\binom{2n}{n}$ to choose a sequence of n integers from $[2n]$ and color them blue. As an example we will use $\{6, 4, 3, 11, 8, 9\}$. The remaining numbers are then $\{1, 2, 5, 7, 10, 12\}$. Next we use each $\left[{n \atop \mu_1, \cdots, \mu_{\ell(\mu)}}\right]_q q^{\sum_{i=1}^{\ell(\mu)} \binom{\mu_i}{2}}$ term to fill in every other space in each brick with integers in descending order and q's as per Theorem 8. We do this first with the blue numbers and then with the black numbers. When we are done we have something like this:

q^5	q^4	q^1	q^2	q^3	q^3	q^2	q^2	q^0	q^0	q^0	q^0
11	10	4	5	9	12	8	7	3	1	6	2

Finally we fill in the x's and -1's as before:

x		1		-1		x		1		1	
q^5	q^4	q^1	q^2	q^3	q^3	q^2	q^2	q^0	q^0	q^0	q^0
11	10	4	5	9	12	8	7	3	1	6	2

Applying the same involution I as before gives us fixed points like this:

x		1		x		x		1		1	
q^5	q^4	q^1	q^2	q^3	q^3	q^2	q^2	q^0	q^0	q^0	q^0
11	10	4	5	9	12	8	7	3	1	6	2

where there are x's each time a $des^{(2)}$ is registered and a 1 over the last pair of a brick. We are also left with two sequences of integers that descend inside each brick with a rise between bricks and inversions labelled. We see then that the sum of the inversions of the two sequences is the definition of $inv^{(2)}(\sigma)$ and so we have that

$$\begin{aligned}\binom{2n}{n}[2n]_q!^2\varphi_q^{(2)}(h_{2n}) &= \sum_{T \text{ a fixed point of } I\in\mathfrak{T}_{\varphi_q}} w(T) \\ &= \sum_{\sigma\in\mathcal{E}_{2n}} x^{des^{(2)}(\sigma)}q^{inv^{(2)}(\sigma)}\end{aligned}$$

Which proves Theorem 7. □

We use this now to show another new generating function. We again use the same argument as in Theorem 5. Let $J_q(u) = \sum_{n\geq 0}\frac{u^n}{[n]_q![n]_q!}q^{2\binom{n}{2}}$. Then

$$\begin{aligned}\sum_{n\geq 0}\frac{t^{2n}}{[2n]_q!^2\binom{2n}{n}}\sum_{\sigma\in\mathcal{E}_{2n}} x^{des^{(2)}(\sigma)}q^{inv^{(2)}(\sigma)} &= \varphi_q^{(2)}\left(\sum_{n\geq 0}h_n t^n\right) \\ &= \varphi_q^{(2)}\left(\frac{1}{\sum_{n\geq 0}e_n(-t)^n}\right) \\ &= \frac{1}{1+\sum_{n\geq 1}\frac{-(x-1)^{n-1}}{[n]_q![n]_q!}q^{2\binom{n}{2}}(t)^{2n}} \\ &= \frac{-1}{-1+\sum_{n\geq 1}\frac{(x-1)^{n-1}}{[n]_q![n]_q!}q^{2\binom{n}{2}}(t)^{2n}} \\ &= \frac{-(x-1)}{-(x-1)+\sum_{n\geq 1}\frac{(x-1)^{n}}{[n]_q![n]_q!}q^{2\binom{n}{2}}(t)^{2n}} \\ &= \frac{-(x-1)}{-(x-1)+(J_q((x-1)t^2)-1)} \\ &= \frac{(1-x)}{-x+J_q((x-1)t^2)}\end{aligned}$$

Using the same method of proof and $inv^{(2)}$ we can also q-analogue $\overline{des^{(2)}}(\sigma)$.

3.4 n-tuples

In this section we will look at places that register a $des^{(2)}(\sigma)$ across multiple permutations. Let $\sigma = \{\sigma^1, \sigma^2, \cdots, \sigma^m\}$. Then, $comdes^{(2)}(\sigma) =$ the number of times $\sigma_i^j > \sigma_i^j + 2$ and $\sigma_i^j + 1 > \sigma_i^j + 3$ for all j. If

$$\begin{aligned}\sigma^1 &= 12\,10\,6\,4\,3\,9\,2\,11\,5\,8\,1\,7 \\ \sigma^2 &= 11\,5\,10\,2\,6\,1\,7\,4\,12\,9\,8\,3 \\ \sigma^3 &= 12\,6\,6\,1\,4\,9\,3\,10\,11\,8\,2\,7\end{aligned}$$

then $comdes^{(2)}(\sigma) = 2$. We also must define a new homomorphism, $\varphi_m^{(2)} : \Lambda_n \to \mathbb{Q}[x]$

$$\begin{aligned} \varphi^{(2)}(e_0) &= 1 \\ \varphi^{(2)}(e_{2n}) &= \frac{(-1)^n(1-x)^{n-1}}{(2n!)^m}\binom{2n}{n}^m = \frac{-(x-1)^{n-1}}{(n!n!)^m} \\ \varphi^{(2)}(e_{2n+1}) &= 0 \end{aligned}$$

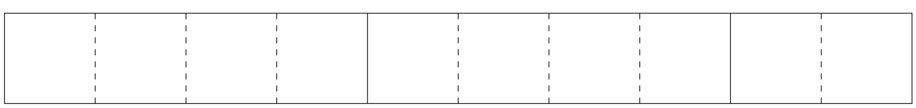
We use these to show the following theorem:

Theorem 9.
$$(2n!)^m \, \varphi^{(2)}(h_{2n}) = \sum_{\sigma \in \mathcal{E}_{2n}^m} x^{comdes^{(2)}(\sigma)}$$

Proof. As before we want to change the LHS into something we can intepret:

$$\begin{aligned} (2n!)^m \, \varphi^{(2)}(h_{2n}) &= (2n!)^m \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} |B_{2\mu,(2n)}| \prod_{i=1}^{\ell(\mu)} \frac{-(x-1)^{\mu_i - 1}}{(2\mu_i!)^m} \binom{2\mu_i}{\mu_i}^m \\ &= (2n!)^m \sum_{\mu \vdash n} (-1)^{2n-\ell(\mu)} |B_{2\mu,(2n)}| (-1)^{\ell(\mu)} (x-1)^{n-\ell(\mu)} \\ &\quad \times \frac{1}{(2\mu_1!)^m (2\mu_2!)^m ... (2\mu_{\ell(\mu)}!)^m} \prod_{i=1}^{\ell(\mu)} \binom{2\mu_i}{\mu_i}^m \\ &= \sum_{\mu \vdash n} |B_{2\mu,(2n)}| \binom{2n}{2\mu_1 2\mu_2 ... 2\mu_{\ell(\mu)}}^m (x-1)^{n-\ell(\mu)} \prod_{i=1}^{\ell(\mu)} \binom{2\mu_i}{\mu_i}^m \end{aligned}$$

We now interpret this starting with $B_{2\mu,(2n)}$ which we use to create a brick tabloid of shape 2μ and size $2n$. For this example let $\mu = (2, 2, 1)$.



We use the term $\binom{2n}{2\mu_1 2\mu_2 ... 2\mu_{\ell(\mu)}}^m$ to choose sets of length $2\mu_i$ out of $[2n]$. We do this m times. Next, we use $\binom{2\mu_i}{\mu_i}$ to color half of the numbers in each $2\mu_i$ set blue and fill in the bricks in descending order with the blue numbers first, every other space in each $2\mu_i$ brick. After the blue numbers we do the black. Again we do this m times. For $m = 2$ such a brick will look like this:

8	9	4	3	7	5	6	1	10	2
10	8	1	6	9	7	3	5	2	4

Finally we put an x or -1 above each pair, with a 1 above the last pair in each brick.

x		1		−1		1		1	
8	9	4	3	7	5	6	1	10	2
10	8	1	6	9	7	3	5	2	4

We apply the involution I as normal with the difference that we are not going to combine two bricks if only some of the m sequences show a $des^{(2)}$ between two bricks. We only combine if a $comdes^{(2)}$ is registered between two bricks. As an example if we have

x		1		x		1		1	
8	9	4	3	10	5	7	2	6	1
10	8	1	6	9	7	3	5	2	4

then applying I gets us

x		1		x		−1		1	
8	9	4	3	10	5	7	2	6	1
10	8	1	6	9	7	3	5	2	4

We can see that if we apply I again we will get what we started with. Fixed points then are those T that have an x above every pair except the last pair in a brick and have no common descents between bricks in all of the m sequences. An example of a fixed point is:

x		1		x		x		1	
8	9	4	3	10	5	7	2	6	1
10	8	1	6	9	7	3	5	2	4

We have then each x of a fixed point registering a *comdes*$^{(2)}$ so the weight of our collection of objects, $\mathcal{T}_{\varphi_m}$, is

$$\sum_{T \text{ a fixed point of } I \in \mathcal{T}_{\varphi_m}} w(T) = \sum_{\sigma \in \mathcal{E}^m_{2n}} x^{comdes^{(2)}(\sigma)}$$

so Theorem 9 is proved. $\square$

This of course will give us another generating function. Let $J^m(u) = \sum_{n\geq 0} \frac{u^n}{(n!n!)^m}$. Proceeding as we did in Theorem 5 again we see that:

$$\begin{aligned}
\sum_{n\geq 0} \frac{t^{2n}}{(2n!)^m} \sum_{\sigma \in \mathcal{E}^m_{2n}} x^{comdes^{(2)}(\sigma)} &= \varphi^{(2)}_m \left(\sum_{n\geq 0} h_n t^n \right) \\
&= \varphi^{(2)}_m \left(\frac{1}{\sum_{n\geq 0} e_n (-t)^n} \right) \\
&= \frac{1}{1 + \sum_{n\geq 1} \frac{-(x-1)^{n-1}}{(n!n!)^m} (t)^{2n}} \\
&= \frac{-1}{-1 + \sum_{n\geq 1} \frac{(x-1)^{n-1}}{(n!n!)^m} (t)^{2n}} \\
&= \frac{-(x-1)}{-(x-1) + \sum_{n\geq 1} \frac{(x-1)^{n}}{(n!n!)^m} (t)^{2n}} \\
&= \frac{-(x-1)}{-(x-1) + (J^m((x-1)t^2) - 1)} \\
&= \frac{(1-x)}{-x + J^m((x-1)t^2)}
\end{aligned}$$

It is also noted that we can combine the q-analogue and n-tuple results. We state here the results obtained but omit the proof as it follows an identical method as before.

Theorem.

$$\binom{2n}{n}^m [2n]_q!^{2m} \varphi^{(2)}_q (h_{2n}) = \sum_{\sigma \in \mathcal{E}^m_{2n}} x^{comdes^{(2)}(\sigma)} q^{\sum_{i=1}^m inv^{(2)}(\sigma_i)}$$

If we let $J^m_q(u) = \sum_{n\geq 0} \frac{u^n}{([n]_q![n]_q!)^m} q^{2m\binom{n}{2}}$ this gives rise to the generating function:

$$\sum_{n\geq 0} \frac{t^{2n}}{[2n]_q!^{2m} \binom{2n}{n}^m} \sum_{\sigma \in \mathcal{E}^m_{2n}} x^{comdes^{(2)}(\sigma)} q^{\sum_{i=1}^m inv^{(2)}(\sigma_i)} = \frac{(1-x)}{-x + J^m_q((x-1)t^2)}$$

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