

FUNCTIONS OF BAIRE CLASS ONE

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ABSTRACT. In this paper, we give a complete characterization of such limits in terms of their continuity points when restricted to closed sets.

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1. INTRODUCTION

Baire-class function is named after Rene Louis Baire (1874-1932) a French mathematician. Baire defined the Baire-class k function by taking the limit of Baire-class $k-1$ with Baire-class 0 is the set of continuous functions (see [1]). He also proved an important result about complete metric spaces which commonly known as the Baire's Category Theorem. In this paper we focus on the space of Baire-class one functions.

We begin by defining Baire's terminology for sets, and the Baire's Category Theorem.

Definition 1.1. *Let X be a metric space. A set $E \subseteq X$ is of first category if it can be written as a countable union of nowhere dense sets, and is of second category if E is not of first category.*

Remark 1.2. The empty set is nowhere dense so is of first category, and hence, a set is of second category must not be empty.

Example 1.3. $\mathbb{Q} = \cup_{x \in \mathbb{Q}} \{x\}$. So, the rational number is of first category and is dense in \mathbb{R} .

Example 1.4. The Cantor set is nowhere dense in \mathbb{R} so it is of first category and is uncountable.

Baire's Category Theorem is an important tool in functional analysis and general topology. We now prove that theorem.

Theorem 1.5 (Baire's Category Theorem).

a) *If (X, d) is a complete metric space and Let $\{U_n\}_1^\infty$ be the sequence of open dense subset of X . Then $\cap_1^\infty U_n$ is dense.*

b) *A complete metric is of second category.*

Proof. a) It suffices to show for any open non-empty set $W \subset X$, we have $W \cap (\cap_1^\infty U_n) \neq \emptyset$. Let $F_n = \cap_{i=1}^n U_i \cap W$ and $B_r(x) = \{y : d(x, y) < r\}$. Since F_1 is open, hence, $\exists r_1 > 0$ and $x_1 \in F_1$ such that $\overline{B_{r_1}(x_1)} \subset F_1$. Now, since F_2 is open in X , so there is a $r_2 < \frac{r_1}{2}$ and $x_2 \in B_{r_1}(x_1) \cap F_2$ such that $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1) \cap F_2$. We can construct a sequence in X inductively by choosing $r_n < \frac{r_{n-1}}{2}$ and $x_n \in B_{r_{n-1}}(x_{n-1}) \cap F_n$ such that $\overline{B_{r_n}(x_n)} \subset B_{r_{n-1}}(x_{n-1}) \cap F_n$. Then $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in X . By completeness of X ,

$x = \lim x_n$ exists and $x \in B_{r_N}(X_N)$ for all N .

Thus $x \in \cap_{n=1}^\infty F_n$

b) Let $\{E_n\}$ be sequence of nowhere dense set. Then $\overline{E_n^c} = O_n$ is open and dense. And $\cap O_n \neq \emptyset$ if and only if $\cup \overline{E_n^c} \neq X$ \square

2. APPLICATION OF BAIRE' CATEGORY THEOREM

We first give a formal definition for Baire-class one functions.

Definition 2.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. f is called Baire-class one if there is a sequence of continuous function converging to f point-wise.*

Of course, this definition can be extended to any metric space as well as any general topological space.

Here is one application for Baire's Category Theorem which the domain is defined on a complete metric space. This theorem and its corollary show that although the point-wise limit of continuous function is not necessarily to be continuous, but its continuity points still form a nice set. [4] has given a proof with the domain defined on the real line. However, since the result for functions that the domain defined on the general complete metric space is more useful especially for our last theorem. Therefore, I stated the theorem for functions that the domain taken on any complete metric space.

Theorem 2.2. *Let X be a complete metric space. If $f : X \rightarrow \mathbb{R}$ is Baire-class one, then the continuity points of f is of second category.*

Proof. Let

$$(1) \quad \text{osc}_x(f) = \limsup_{y \rightarrow x} f(y) - \liminf_{y \rightarrow x} f(y)$$

be the oscillation of f at x . It is clear that f continuous at x if and only if $\text{osc}_x(f) = 0$. Let $D = \cup_{n=1}^{\infty} D_n$ be the set of discontinuity points of f where $D_n = \{x : \text{osc}_x(f) \geq \frac{1}{n}\}$ is a closed set. So it suffices to show D_n^c is dense in X . Suppose not, then there is an open set I such that $I \cap D_n^c = \emptyset$ and hence $\bar{I} \cap D_n^c = \emptyset$

Consider the set $E_k = \cap_{i,j \geq k} \{x : |f_j(x) - f_i(x)| \leq \frac{1}{4n}\}$. Then E_k is closed, $E_1 \subseteq E_2 \subseteq \dots$ and their union is X , so E_k is not nowhere dense and $\bar{I} = \cup_1^{\infty} (\bar{I} \cap E_k)$. So, Baire's theorem implies $E_k \cap \bar{I}$ is not nowhere dense in \bar{I} for some k . So, there is an open set $J \subset E_k \cap \bar{I}$ and $|f_j(x) - f_i(x)| \leq \frac{1}{4n}$ for all $x \in J$ and $i, j \geq k$. Thus, by letting $i \rightarrow \infty$ we have $|f(x) - f_k(x)| \leq \frac{1}{4n}$. Since f_k is continuous, so there is an open set $O \subseteq J$ such that $|f_k(y) - f_k(x)| \leq \frac{1}{4n} \forall x, y \in O$. Thus $\forall x, y \in O$ we have $|f(x) - f(y)| \leq \frac{3}{4n} < \frac{1}{n}$, but $O \cap D_n^c = \emptyset$. Thus D_n^c is dense. \square

Corollary 2.3. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is Baire-class one, then its point of continuity form a dense set.*

Proof. D_n is closed and nowhere dense. Thus D_n^c is open and dense. So, the result follows from Baire's category theorem. \square

Corollary 2.4. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, then its derivative continuous at a dense set of points*

There are more applications for the Baire's Category Theorem (see [2] and [3]).

3. UNIFORM LIMIT OF BAIRE-CLASS ONE FUNCTIONS

The following proposition shows the space of Baire-class one functions is closed under the uniform limit. Also, the reader can find further developments of this proposition in [5].

Proposition 3.1. *The uniform Limit of Baire-class one functions is Baire-class one.*

Proof. Let $f_n \rightarrow f$ uniformly, each f_n is Baire-class one. Then there is a subsequence $g_k = f_{n_k}$ such that $|g_k - f| < \frac{1}{2^k}$. Then we can rewrite the limiting function $f(x) = g_1(x) + \sum_{n=2}^{\infty} (g_n(x) - g_{n-1}(x))$, and $|g_n - g_{n-1}| < \frac{3}{2^n} \forall n$. So, it suffices to show $\sum_2^{\infty} (g_n(x) - g_{n-1}(x))$ is Baire-class one.

Let $\phi_{n,k}$ be continuous and converges point-wise to $(g_n - g_{n-1})$ as $k \rightarrow \infty$. We can assume $|\phi_{n,k}| \leq \frac{3}{2^n} \forall k$ since otherwise define $\tilde{\phi}_{n,k} = \phi_{n,k} 1_{\{x: |\phi_{n,k}(x)| \leq \frac{3}{2^n}\}} + \frac{3}{2^n} 1_{\{x: \phi_{n,k}(x) > \frac{3}{2^n}\}} - \frac{3}{2^n} 1_{\{x: \phi_{n,k}(x) < -\frac{3}{2^n}\}}$. So, $h_k = \sum_{n=2}^{\infty} \phi_{n,k}$ make sense and the convergence for this infinite sum is uniform. Now fix x and let ϵ be given, we can choose an N large such that $\sum_N^{\infty} |(g_n - g_{n-1})|$ and $\sum_{n=N}^{\infty} |\phi_{n,k}|$ both $< \epsilon$, then

$$\begin{aligned}
(2) \quad & |h_k(x) - \sum_2^{\infty} (g_n(x) - g_{n-1}(x))| \\
(3) \quad & \leq \sum_{n=N}^{\infty} |g_n - g_{n-1}| + \sum_{n=N}^{\infty} |\phi_{n,k}| \\
(4) \quad & + \left| \sum_{n=2}^{N-1} \phi_{n,k}(x) - \sum_{n=2}^{N-1} (g_n(x) - g_{n-1}(x)) \right| \\
(5) \quad & < 2\epsilon \text{ as } k \rightarrow \infty
\end{aligned}$$

Finally, we let $\epsilon \rightarrow 0$ and the proof is complete. \square

However, it may not be preserved under the point-wise limit. Here is a counter example.

Example 3.2. [2] Consider the indicator function of the rational number $1_{\mathbb{Q}}$. Also consider $f_{m,n}(x) = |\cos(m!\pi x)|^n$. Then $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} f_{m,n}(x) = 1_{\mathbb{Q}}(x)$. Thus $1_{\mathbb{Q}}$ is pointwise limit of Baire-class one, but it has no point of continuity so is not Baire-class one function.

4. PROPERTIES OF BAIRE-CLASS ONE FUNCTIONS

In this section we will study some properties of Baire-class one functions

Definition 4.1. Let X be a metric space. $A \subseteq X$ is G_{δ} if it is countable intersection of open sets, and is F_{σ} if it is countable union of closed sets

Proposition 4.2. If f is Baire-class one, then $f^{-1}(a, \infty)$ and $f^{-1}(-\infty, a)$ are F_{σ} for all $a \in \mathbb{R}$.

Proof. It suffices to show $f^{-1}(a, \infty)$ is F_{σ} . Let $\{f_n\}$ be sequence of continuous functions that converges to f point-wise. We claim that

$$(6) \quad f^{-1}(a, \infty) = \cup_{m=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n \geq k}^{\infty} f_n^{-1}\left[a + \frac{1}{m}, \infty\right)$$

If $x \in f^{-1}(a, \infty)$. Since $\lim f_n(x) = f(x)$, hence for sufficiently large k , we have $a + \frac{1}{m} \leq f_n(x)$ for some m and for all $n \geq k$. Thus, $x \in \cap_{n \geq k}^{\infty} f_n^{-1}\left[a + \frac{1}{m}, \infty\right)$. Conversely, if $x \in \cup_{m=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n \geq k}^{\infty} f_n^{-1}\left[a + \frac{1}{m}, \infty\right) \Rightarrow x \in \cap_{n \geq k}^{\infty} f_n^{-1}\left[a + \frac{1}{m}, \infty\right)$ for some m and k
 $\Rightarrow a < a + \frac{1}{m} \leq \lim f_n(x) = f(x)$. \square

In fact, the converse of this proposition is also true. The proof needs the following lemmas.

Lemma 4.3. Let A and B be two disjoint closed sets in \mathbb{R} , then there is a continuous function f that is 1 on A and 0 on B with $0 \leq f \leq 1$

Proof. A and B are disjoint, so we can define a function f such that $f|_A = 1$ and $f|_B = 0$. Now, $A^c \cap B^c$ is open in \mathbb{R} , so it can be written as countable union of disjoint open intervals (a_n, b_n) . On each $[a_n, b_n]$, We define

$$(7) \quad f(x) = \begin{cases} 1 & \text{if } a_n, b_n \in A \\ 0 & \text{if } a_n, b_n \in B \\ \text{linear} & \text{if } a_n, b_n \text{ are in different set} \end{cases}$$

□

Lemma 4.4. $F \subseteq \mathbb{R}$ is both F_σ and G_δ if and only if 1_F , the indicator function of F , is Baire-class one.

Proof. F is both F_σ and G_δ , then F^c is F_σ . So $F = \cup_1^\infty A_n$ and $F^c = \cup_1^\infty B_n$, A_n and B_n are closed for all n. Moreover $A_n \cap B_m = \emptyset$ for all n, m . We may assume A_n and B_n are increasing (i.e. $A_1 \subseteq A_2 \subseteq \dots$). By lemma 4.3, for each n, there is a continuous function f_n such that 1 on A_n and 0 on B_n . Then $f_n \rightarrow 1_F$ points-wise. The converse is clear from proposition 4.2. and the fact $f^{-1}[a, \infty) = f^{-1}(a, \infty)$ for all $a \in (0, 1)$ □

Lemma 4.5. Let A and B be two closed set in \mathbb{R} , then $A \setminus B$ is F_σ

Proof. $A \setminus B = A \cap B^c$, and B^c is open set in \mathbb{R} , so is F_σ . So, the result follows from deMorgan's Law. □

Lemma 4.6. Let A and B be F_σ sets in \mathbb{R} . Then there exists A', B' F_σ -set such that $A' \subseteq A$ and $B' \subseteq B$, $A \cup B = A' \cup B'$, and $A' \cap B' = \emptyset$.

Proof. Let $A = \cup_1^\infty A_n, B = \cup_1^\infty B_n$. We may assume A_n, B_n are increasing for n . Let $C_n = (B_n \cap A_n) \setminus (B_{n-1} \cup A_{n-1})$, then C_n is F_σ for all n. So, we let $A'_n = A_n \setminus B_n$, and $B'_n = (B_n \setminus A_n) \cup C_n$ and set $A' = \cup_1^\infty A'_n$ and $B' = \cup_1^\infty B'_n$ □

Remark 4.7. Let A, B and C be F_σ , we let $D = A \cup B$, then there is a $D' \subseteq D$ and $C' \subseteq C$ F_σ such that $D' \cup C' = D \cup C$ and $D' \cap C' = \emptyset$. Also, there exist $A' \subseteq A \cap D'$ and $B' \subseteq B \cap D'$ F_σ and $A' \cup B' = D'$ with $A' \cap B' = \emptyset$. Then $A \cup B \cup C = A' \cup B' \cup C'$ and they are F_σ and pairwise disjoint with $A' \subseteq A$, $B' \subseteq B$, and $C' \subseteq C$.

Remark 4.8. We can apply the same construction inductively to any finite union of F_σ set.

Now, we are ready to prove the converse of Proposition 4.2.

Proposition 4.9. if $f^{-1}(a, \infty)$ and $f^{-1}(-\infty, a)$ are F_σ for all $a \in \mathbb{R}$, then f is Baire-class one.

Proof. By composing with $\tilde{f} = \frac{1}{2}(1 + \tanh(f))$. We may assume $0 \leq f \leq 1$. Now, $f^{-1}(a, b)$ is F_σ for all $a < b \in \mathbb{R}$. So, for $n \in \mathbb{N}$, let

$$(8) \quad E_k^n = f^{-1}(k2^{-n}, (k+1)2^{-n} + 2^{-2n})$$

Then there are $G_k^n \subseteq E_k^n$ F_σ -set such that $G_k^n \cap G_l^n = \emptyset \forall m \neq l$, and $\cup_{k=1}^n G_k^n = \mathbb{R}$. So, consider the function

$$(9) \quad \phi_n = \sum_{k=0}^{2^n-1} k2^{-n} 1_{G_k^n}$$

Write $E_k^n = f^{-1}(k2^{-n}, (k+1)2^{-n}] \cup (E_k^n \cap E_{k+1}^n)$. If x is in the first part, then $|\phi_n(x) - f(x)| \leq \frac{1+2^{-n}}{2^n}$. If x is in the second part, then x is in either G_k^n or G_{k+1}^n , then $|f(x) - \phi_n(x)| \leq \frac{1+2^{-n}}{2^n}$. So, ϕ_n converges to f uniformly and each ϕ_n are Baire-class one. So, f is Baire-class one. \square

5. THE PRINCIPLE OF TRANSFINITE INDUCTION

The principle of transfinite induction is a crucial tool for the last and the main result on this paper, and also is a useful technique in many aspect in mathematics. [6] has more discussion. We begin by defining the well ordered set.

Definition 5.1. *If I is linearly ordered by $<$ and every subset of I has a unique minimal element in I , I is said to be well ordered by $<$.*

Theorem 5.2 (The Principle of Transfinite Induction). *Let I be well order set. Define $I_x = \{y \in I : y < x\}$ be the set of predecessor of x . If $A \subset I$ and if $I_x \subset A$ implies $x \in A$. Then $A = I$*

Proof. If $I \neq A$ let $x = \inf(I \setminus A)$. Then $I_x \subset A$ but $x \notin A$. \square

So, if the well ordered set is the nature number, then it is just the ordinary induction. We also use the transfinite recursion in the following way (see [6]).

Theorem 5.3 (The Principle of Transfinite Recursion). *Let I be well ordered set and S be a set. Let \mathcal{F} be the set of all functions $f : I_x \rightarrow S$. Then given a function $G : \mathcal{F} \rightarrow S$, then there is a unique $f : I \rightarrow S$ such that $f(x) = G(f|_{I_x})$.*

In practice this just means that if we have partially defined a function $f : I \rightarrow S$ for all $\alpha < \beta$, and given this we have a rule for the definition of f at β , then we can define f for all $\alpha \in I$.

We now ready to prove a useful consequence in the real line.

Proposition 5.4. *Let I be well ordered set and $\{\mathcal{C}_\alpha\}$ is a collection of closed set in \mathbb{R} indexed by $\alpha \in I$ such that they are nested by reverse inclusion which is $\mathcal{C}_\alpha \subsetneq \bigcap_{\beta < \alpha} \mathcal{C}_\beta$. Then there are at most countably many \mathcal{C}_α*

Proof. Let $\mathcal{O}_\alpha = \mathcal{C}_\alpha^c$, then each \mathcal{O}_α are open set. Let $U = \{U_i\}$ be countable base for \mathbb{R} (i.e. $U_i = B_{\frac{1}{n}}(x)$, $x \in \mathbb{Q}$). If \mathcal{O} is open and $x \in \mathcal{O}$, then there is a U_i such that $x \in U_i \subseteq \mathcal{O}$.

Define $f : I \rightarrow U$ by

- 1) Let $\alpha_0 = \min(I)$ and take $x_{\alpha_0} \in U_{\alpha_0} \subseteq \mathcal{O}_{\alpha_0}$ and set $f(\alpha_0) = U_{\alpha_0}$
- 2) If we have defined U_α for all $\alpha < \beta$ take $x_\beta \in \mathcal{O}_\beta \setminus \bigcup_{\alpha < \beta} \mathcal{O}_\alpha$ then there is a U_β such that $x_\beta \in U_\beta \subseteq \mathcal{O}_\beta$ and set $f(\beta) = U_\beta$

So, f exists by transfinite recursion. We will show f is injective by transfinite induction: If we assume f is injective for all $\alpha < \beta$, then by construction it is clear that $f(\beta) \neq f(\alpha)$ for all $\alpha < \beta$ because $x_\beta \in f(\beta)$ while $x_\beta \notin f(\alpha)$ for all $\alpha < \beta$. \square

6. BAIRE'S CHARACTERIZATION THEOREM

Now we prove our last result.

Theorem 6.1 (Baire's Characterization Theorem). *A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is Baire-class one iff for every closed subset $S \subseteq \mathbb{R}$ there exists a point $x \in S$ such that $f|_S$ is continuous at x with respect to the subspace topology (i.e. for every sequence $x_n \in S$ with $x_n \rightarrow x$ one has $f(x_n) \rightarrow f(x)$).*

Proof. \Leftarrow) Suppose for every closed subset $S \subseteq \mathbb{R}$ there exists a point $x \in S$ such that $f|_S$ is continuous at x . By proposition 4.9 It suffices to show $f^{-1}(a, \infty)$ and $f^{-1}(-\infty, a)$ are F_σ . First, we will show $f^{-1}(a, \infty)$ is F_σ . Let $a < b$ and \mathcal{I} be a well ordered set. Let $0 = \min(\mathcal{I})$ and $C_0 = \mathbb{R}$, by the assumption there is an $x_0 \in C_0$ such that f is continuous at x_0 . so, there is an open set I_0 such that if $f(x_0) > a$ then $f(I_0) \subseteq (a, \infty)$ or if $f(x_0) < b$ then $f(I_0) \subseteq (-\infty, b)$. We divide it into two cases,

- a) Set $A_0 = I_0$ if $f(x_0) > a$ and $A_0 = \emptyset$ otherwise.
- b) Set $B_0 = I_0$ if $f(x_0) \leq a$ and $B_0 = \emptyset$ otherwise.

Then A_0 and B_0 are F_σ

We can continuous the process with transfinite recursion. If for all $\beta \in \mathcal{I}_\alpha$ have been defined then set $C_\alpha = \mathbb{R} \setminus ((\cup_{\beta < \alpha} I_\beta)$, then pick a x_α which $f|_{C_\alpha}$ continuous at x_α . Choose an open set I_α relative to C_α as above and define

- a) $A_\alpha = I_\alpha$ if $f(x_\alpha) > a$ and $A_\alpha = \emptyset$ otherwise.
- b) $B_\alpha = I_\alpha$ if $f(x_\alpha) \leq a$ and $B_\alpha = \emptyset$ otherwise.

Then A_α and B_α are F_σ in \mathbb{R} .

We end the process when C_α reaches the empty set. Now, $\{C_\alpha\}$ is well ordered with reverse inclusion so is countable by proposition 5.4 , and each A_α and B_α is F_σ set and so is $A = \cup A_\alpha$ and $B = \cup B_\alpha$ with $A \subseteq f^{-1}(a, \infty)$ and $B \subseteq f^{-1}(-\infty, b)$. Thus, $f^{-1}[b, \infty) \subseteq A \subseteq f^{-1}(a, \infty)$. Since b can be chosen arbitrary, by taking a sequence $b_n > a$ and converges to a . So, $f^{-1}(a, \infty)$ is an F_σ set. And apply the same process to $f^{-1}(-\infty, a)$ but with $b < a$ to conclude $f^{-1}(-\infty, a)$ is F_σ .

\Rightarrow) It is clear from theorem 2.2. \square

The closedness condition of the restriction set is crucial. In fact there is a counter example for which the function is Baire class one, but it has no point of continuity when it restricted on the the of rational number. We end this paper with the following example.

Example 6.2. *Consider the rational ruler function that is*

$$(10) \quad f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ } p \in \mathbb{N} \text{ and } q \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Let $S \subseteq \mathbb{R}$ be closed. Since f is continuous on \mathbb{Q}^c , so it suffices to show $S \subset \mathbb{Q}$, $f|_S$ has a point of continuity, but then S is not a perfect set which implies there is an isolated point $x \in S$, and $f|_S$ continuous at x . Thus, f is Baire class one. But there is no continuity points when f restricted on \mathbb{Q} . Since $\frac{p(r-1)}{qr} \rightarrow \frac{p}{q}$ as $r \rightarrow \infty$ but $f(\frac{p(r-1)}{qr}) = \frac{1}{qr} \rightarrow 0$ for any $\frac{p}{q}$

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