

On the Fundamental Groups of Ricci Solitons

Nathan Chen

Advisor: Prof. Bennett Chow

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Department of Mathematics
University of California, San Diego

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1 Introduction

In the early 1900s, Henri Poincaré proposed a topological problem that would later be known as the *Poincaré conjecture*. We usually see the standard phrasing of the conjecture as follows:

Every simply connected closed 3-manifold is homeomorphic to the 3-sphere.

By *simply connected*, we mean that any closed path on the manifold can be continuously shrunk to a point, while staying in the manifold.

Later on, in the 1980s, William Thurston presented a more general problem, known as the *geometrization conjecture*, which stated that any closed 3-manifold can be divided into pieces that fall into one of eight possible geometric structures called Thurston geometries. At around the same time, Richard Hamilton introduced Ricci flow, which was a process that shaped the metric of a manifold. Using Ricci flow, Grigori Perelman was able to prove the geometrization conjecture in 2003, which in turn implied the Poincaré conjecture.

In this paper, our aim is to introduce the Ricci soliton equation and study a recently proved result by William Wylie about the fundamental group of Riemannian manifolds that satisfy one version of the Ricci soliton equation.

2 Notation and Background

Unless otherwise stated, we will denote (\mathcal{M}, g) to be a connected, oriented, and complete Riemannian manifold with a positive-definite metric g . In other words, \mathcal{M} cannot be rewritten as a union of two or more disjoint nonempty open subsets (connected), and every Cauchy sequence on \mathcal{M} converges to a point in \mathcal{M} . Intuitively, the latter part means that \mathcal{M} is not missing any points inside or on the boundary of \mathcal{M} . We may also use \mathcal{M}^n when specifying that the manifold is n -dimensional. Let $T(\mathcal{M})$ be its tangent bundle, and let $T_p\mathcal{M}$ denote the tangent plane to \mathcal{M} at a point $p \in \mathcal{M}$. $\chi(\mathcal{M})$ will represent the space of smooth vector fields on \mathcal{M} . We will often use g in place of the inner product $\langle \cdot, \cdot \rangle$ when emphasizing the metric. It should be noted that much of the material in this paper has been derived from the references on the last page.

2.1 Connections

Recall that a *connection* $\nabla : \chi(\mathcal{M}) \times \chi(\mathcal{M}) \rightarrow \chi(\mathcal{M})$ is a bilinear map such that for all $X, Y \in \chi(\mathcal{M})$ and for any smooth function f on M , the following two properties are satisfied:

$$\nabla_{fX}Y = f\nabla_XY,$$

$$\nabla_X fY = f\nabla_X Y + X(f)Y.$$

$\nabla_X Y$ is called the covariant derivative of Y in the direction of X .

From the Fundamental Theorem of Riemannian Geometry, we know that for every Riemannian manifold (\mathcal{M}, g) , there exists a unique connection ∇ (called the *Levi-Civita connection*) with the following two conditions:

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Here, $X\langle Y, Z \rangle$ signifies that we start with the inner product of two vector fields Y, Z and then differentiate in the X direction. In addition, $[,]$ represents the Lie bracket, which is defined by $[X, Y]f = (XY - YX)f$. In general, we say that a connection is *torsion-free* if it satisfies the second property.

2.2 Pullback and Pushforward

Let $\phi : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map, and let X be a vector field on \mathcal{M} . Then X assigns to each point $p \in \mathcal{M}$ a tangent vector $X_p \in T_p(\mathcal{M})$. For each point $p \in \mathcal{M}$, we can *push forward* X_p to get a vector $\phi_* X_p \in T_{\phi(p)}\mathcal{N}$. In general, we cannot push forward X to obtain a vector field over \mathcal{N} because ϕ is not necessarily surjective or injective.

Building upon the pushforward map $\phi_* : T_p\mathcal{M} \rightarrow T_{\phi(p)}\mathcal{N}$, we arrive at a dual linear map

$$(\phi_*)^* : T_{\phi(p)}^*\mathcal{N} \rightarrow T_p^*\mathcal{M};$$

we call this the *pullback* of ϕ , and to simplify notation, we write ϕ^* in place of $(\phi_*)^*$. Here, $T_p^*\mathcal{M}$ is the dual space to $T_p\mathcal{M}$ (we call it the *cotangent space* at p). Now suppose that $\omega \in T_{\phi(p)}^*\mathcal{N}$ is a covector and we have a vector $Y \in T_p\mathcal{M}$. Then by definition,

$$(\phi^*\omega)(Y) = \omega(\phi_*Y).$$

In contrast to how the pushforward behaves, one can show that smooth covector fields always pull back to smooth covector fields.

In the special case where $\mathcal{M} = \mathcal{N}$, the pushforward corresponds to a change in coordinates for a vector field. The pullback would correspond to switching differentials from one coordinate system to another by applying the chain rule (for example, when integrating in polar vs cartesian coordinates).

2.3 First Variation of Arclength

Let $\gamma : [a, b] \rightarrow \mathcal{M}$ be a piecewise smooth curve. By definition, the length of γ , denoted $L(\gamma)$, is given by

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

Proposition 1: The length of γ is independent of parametrization.

Proof. Let $\bar{\gamma}$ be a reparametrization of γ . Then there exists a diffeomorphism $\alpha : [c, d] \rightarrow [a, b]$ such that $\bar{\gamma} = \gamma \circ \alpha$. Furthermore, assume that α is a regular curve, such that $\alpha'(t) \neq 0$ for all $t \in [c, d]$. Then without loss of generality assume that $\alpha'(t) > 0$ (If $\alpha'(t) < 0$, then we have two sign changes in the calculations, which cancel out). Then

$$\begin{aligned} L(\bar{\gamma}) &= \int_c^d \|\bar{\gamma}'(t)\| dt = \int_c^d \left\| \frac{d}{dt}(\gamma \circ \alpha)(t) \right\| dt \\ &= \int_c^d \|\gamma'(\alpha(t))\alpha'(t)\| dt \\ &= \int_c^d \|\gamma'(\alpha(t))\| \alpha'(t) dt, \end{aligned}$$

and using the change of variables $\alpha(t) = s$ (so that $\alpha'(t)dt = ds$), it follows that

$$L(\bar{\gamma}) = \int_a^b \|\gamma'(s)\| ds = L(\gamma). \quad \square$$

Let $p, q \in \mathcal{M}$; if we define the distance $d(p, q)$ between the two points to be the infimum of the lengths of all curves from p to q , then M becomes a metric space.

Let $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$ be smooth, for some $\varepsilon > 0$. Define $\varphi(t, s) = \gamma_s(t)$, such that

$$\varphi|_{[a, b] \times \{0\}} = \gamma_0 : [a, b] \rightarrow \mathcal{M};$$

we call φ a *smooth variation* of γ_0 . Let T and V be the fields of tangent vectors on $[a, b] \times (-\varepsilon, \varepsilon)$ with respect to the first and second variables. Then

$$T = \varphi_* \left(\frac{\partial}{\partial t} \right), \quad V = \varphi_* \left(\frac{\partial}{\partial s} \right).$$

Assume that γ is a smooth and $\|\gamma'(t)\| \neq 0$. We may also assume without loss of generality that γ is parametrized proportional to arclength when $s = 0$, i.e. $\|\gamma'_0(t)\| = c$, a constant.

The first variation formula describes the rate of change in arclength over the family of curves γ_s , where $s \in (-\varepsilon, \varepsilon)$. We start with

$$\begin{aligned} \frac{d}{ds} L(\gamma_s) &= \frac{d}{ds} \int_a^b \|\gamma'_s(t)\| dt = \frac{d}{ds} \int_a^b \langle \gamma'_s(t), \gamma'_s(t) \rangle^{1/2} dt \\ &= \int_a^b \frac{d}{ds} \langle T, T \rangle^{1/2} dt \\ &= \int_a^b \frac{1}{2} \langle T, T \rangle^{-1/2} \cdot \frac{d}{ds} (\langle T, T \rangle) dt \\ &= \frac{1}{2} \int_a^b \|T\|^{-1} \cdot V(\langle T, T \rangle) dt \\ &= \int_a^b \|T\|^{-1} \cdot \langle \nabla_V T, T \rangle dt. \end{aligned}$$

Using the fact that $[T, V] = \nabla_T V - \nabla_V T = [\frac{\partial}{\partial t}, \frac{\partial}{\partial s}]\varphi = 0$ (because partial derivatives commute), we have

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = \frac{1}{c} \int_a^b \langle \nabla_T V, T \rangle dt,$$

where $\|T\| = c$ at $s = 0$. By integrating the equation

$$\frac{\partial}{\partial t} \langle V, T \rangle = T \langle V, T \rangle = \langle \nabla_T V, T \rangle + \langle V, \nabla_T T \rangle,$$

it follows that

$$\langle V, T \rangle|_a^b = \int_a^b \langle \nabla_T V, T \rangle dt + \int_a^b \langle V, \nabla_T T \rangle dt.$$

Hence, we obtain the first variation formula

$$\frac{d}{ds}L(\gamma_s)|_{s=0} = \frac{1}{c} \left(\langle V, T \rangle|_a^b - \int_a^b \langle V, \nabla_T T \rangle dt \right).$$

Recall that intuitively, a *geodesic* is a curve that is locally length-minimizing. In other words, the (covariant) derivative of the tangent vector along the curve should vanish. This leads us to the following definition.

Definition: A geodesic on \mathcal{M} is a curve $\gamma(t)$ such that $\nabla_{\gamma'}\gamma' = 0$.

In section 2.1, we defined a connection to be a map that takes in vector fields. Since γ' only gives a vector field along a curve, we must extend γ' to a smooth neighborhood of γ (the resulting vector field extension V can be arbitrary), apply the covariant derivative of V in the direction of V , and then restrict it to the curve. In fact, the extended equation gives values that only depend on the curve.

2.4 Riemann Curvature Tensor

Definition: The *Riemann curvature tensor* identifies a point $p \in \mathcal{M}$ with a trilinear map $\text{Rm} : T_p\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \rightarrow T_p\mathcal{M}$ such that if X, Y, Z are vector field extensions of vectors $x, y, z \in T_p\mathcal{M}$, respectively, then

$$\text{Rm}(x, y)z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

In fact, $\text{Rm}(x, y, z)$ does not depend on the choice of the vector field extension. By computation, Rm satisfies the *First Bianchi identity*:

$$\text{Rm}(x, y)z + \text{Rm}(y, z)x + \text{Rm}(z, x)y = 0.$$

In its present form, we have defined the Riemannian curvature as a $(3, 1)$ -tensor. We can obtain a 4-tensor by simply writing

$$\text{Rm}(x, y, z, w) = g(\text{Rm}(x, y)z, w).$$

A related notion is *sectional curvature*.

Definition: Fix $p \in \mathcal{M}$, and let $P \subset T_p\mathcal{M}$ be a 2-dimensional plane with an orthonormal basis $\{e_1, e_2\}$. Then we define the sectional curvature of P to be

$$K(P) = g(\text{Rm}(e_1, e_2)e_2, e_1).$$

The brief computation below shows that $K(P)$ is independent of the choice of orthonormal basis, i.e. it only depends on P ; suppose

$$\{a_1 = ce_1 + de_2, a_2 = -de_1 + ce_2\}$$

is another orthonormal basis for P , where $c = \cos(\theta)$ and $d = \sin(\theta)$ for some angle θ . Then by the symmetry of Rm ,

$$\begin{aligned} g(\text{Rm}(a_1, a_2)a_2, a_1) &= g(\text{Rm}(ce_1 + de_2, -de_1 + ce_2)(-de_1 + ce_2), ce_1 + de_2) \\ &= (c^2 - (-d^2))g(\text{Rm}(e_1, e_2)(-de_1 + ce_2), ce_1 + de_2) \\ &= (c^2 + d^2)^2g(\text{Rm}(e_1, e_2)(e_2), e_1) \\ &= g(\text{Rm}(e_1, e_2)e_2, e_1) \end{aligned}$$

2.5 Second Variation of Arclength

Assume that $\gamma : [a, b] \rightarrow M$ is a geodesic. Let $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \rightarrow \mathcal{M}$ be smooth, for some $\varepsilon, \delta > 0$, and define $\varphi(t, v, u) = \gamma_{u,v}(t)$ such that $\varphi(t, 0, 0) = \gamma(t)$ and φ is a 2-parameter variation of γ . If $L(\gamma_{v,u})$ denotes the arclength of the curve that sends $t \mapsto \varphi(t, v, u)$, then by definition

$$L(\gamma_{v,u}) = \int_a^b \|T\| ds.$$

Assume that γ is parametrized by arclength, so that $\|\gamma'\| = 1$. Similar to above, let

$$T = \varphi_*\left(\frac{\partial}{\partial t}\right), \quad V = \varphi_*\left(\frac{\partial}{\partial v}\right), \quad U = \varphi_*\left(\frac{\partial}{\partial u}\right)$$

be the fields of tangent vectors on $[a, b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta)$ with respect to t, v , and u . From the proof of the first variation formula, recall that

$$\frac{\partial}{\partial v}L(\gamma_{v,u}) = \int_a^b \|T\|^{-1} \cdot \langle \nabla_T V, T \rangle ds.$$

Differentiating both sides with respect to u , we see that

$$\begin{aligned} \frac{\partial^2}{\partial u \partial v}L(\gamma_{v,u}) &= \int_a^b \frac{\partial}{\partial u} \left(\|T\|^{-1} \cdot \langle \nabla_T V, T \rangle \right) ds \\ &= \int_a^b \frac{\partial}{\partial u} \left(\langle T, T \rangle^{-1/2} \right) \cdot \frac{\partial}{\partial u} \left(\langle T, T \rangle \right) \cdot \langle \nabla_T V, T \rangle \end{aligned}$$

$$\begin{aligned}
& + \|T\|^{-1} \cdot U \left(\langle \nabla_T V, T \rangle \right) ds \\
& = \int_a^b -\langle T, T \rangle^{-3/2} \cdot \langle \nabla_U T, T \rangle \cdot \langle \nabla_T V, T \rangle \\
& \quad + \|T\|^{-1} \cdot (\langle \nabla_U \nabla_T V, T \rangle + \langle \nabla_T V, \nabla_U T \rangle) ds \\
& = \int_a^b -\|T\|^{-3} \cdot \langle \nabla_T U, T \rangle \cdot \langle \nabla_T V, T \rangle \\
& \quad + \|T\|^{-1} \cdot (\langle \nabla_U \nabla_T V, T \rangle + \langle \nabla_T V, \nabla_U T \rangle) ds.
\end{aligned}$$

We can rewrite this integral in terms of the Riemann curvature tensor using the definition

$$\text{Rm}(U, T)V = \nabla_U \nabla_T V - \nabla_T \nabla_U V - \nabla_{[U, T]}V.$$

Using the fact that $[U, T] = 0$ because the partial derivatives commute,

$$\begin{aligned}
\frac{\partial^2}{\partial u \partial v} L(\gamma_{v,u}) & = \int_a^b -\|T\|^{-3} \cdot \langle \nabla_T U, T \rangle \cdot \langle \nabla_T V, T \rangle \\
& \quad + \|T\|^{-1} \cdot (\langle \text{Rm}(U, T)V, T \rangle + \langle \nabla_T \nabla_U V, T \rangle + \langle \nabla_T V, \nabla_U T \rangle) ds.
\end{aligned}$$

Since γ is a geodesic that is parametrized by arclength, $\|T\| \big|_{(0,0)} = 1$ and $\nabla_T T \big|_{(0,0)} = 0$. Hence, we arrive at the second variation formula:

$$\begin{aligned}
\frac{\partial^2}{\partial u \partial v} L(\gamma_{v,u}) \big|_{(0,0)} & = \int_a^b \langle \text{Rm}(U, T)V, T \rangle + \langle \nabla_T \nabla_U V, T \rangle + \langle \nabla_T V, \nabla_U T \rangle \\
& \quad - \langle \nabla_T U, T \rangle \cdot \langle \nabla_T V, T \rangle ds. \\
& = \int_a^b -T \langle \nabla_U V, T \rangle - \langle \text{Rm}(U, T)T, V \rangle + \langle \nabla_T V, \nabla_T U \rangle \\
& \quad - T \langle U, T \rangle \cdot T \langle V, T \rangle ds. \\
& = -\langle \nabla_U V, T \rangle \big|_a^b - \int_a^b \langle \nabla_T V, \nabla_T U \rangle ds \\
& \quad + \int_a^b \langle \text{Rm}(U, T)T, V \rangle - T \langle U, T \rangle \cdot T \langle V, T \rangle ds.
\end{aligned}$$

Now let us define

$$(\nabla_T V)^\perp = \nabla_T V - \langle \nabla_T V, T \rangle T$$

as the projection of $\nabla_T V$ onto an orthonormal vector field to T .

Corollary: Fix two points p and q . Suppose we revert back to a 1-parameter family γ_v of piecewise smooth paths from p to q such that $\gamma(0)$ is a geodesic parametrized by arclength. Then we can rewrite the second variation of arclength as

$$\frac{\partial^2}{\partial v^2} L(\gamma_v) \big|_{v=0} = \int_a^b \left(\left| (\nabla_T V)^\perp \right|^2 - \langle \text{Rm}(V, T)T, V \rangle \right) ds. \quad (2.1)$$

2.6 Covering Maps

Let us now touch on a topic that will be used later on.

Definition: Let X, \tilde{X} be topological spaces. A *covering map* $\pi : \tilde{X} \rightarrow X$ must satisfy the following properties: (1) \tilde{X} is path connected and locally path connected, (2) π is surjective and continuous, and (3) $\forall p \in X$, there exists a neighborhood U of p such that U is connected and each component of $\pi^{-1}(U)$ is mapped homeomorphically onto U via π . Then we say that U is *evenly covered* by π , and \tilde{X} is a *covering space* of X .

Proposition: If X is a connected and locally simply connected topological space (i.e. X admits a basis of simply connected open sets), then there exists a simply connected topological space \tilde{X} and a covering map $\pi : \tilde{X} \rightarrow X$; these are unique up to homeomorphism. \tilde{X} under this proposition is called the *universal covering space* of X .

Definition: A *covering transformation* (also known as a *deck transformation*) of π is a homeomorphism $\varphi : \tilde{X} \rightarrow \tilde{X}$ such that $\pi \circ \varphi = \pi$. The group of deck transformations of the universal cover π is isomorphic to the fundamental group $\pi_1(X)$.

2.7 Fundamental Group

Definition: Let $I = [0, 1]$ be an interval, and let X be a topological space. Two paths $f, g : I \rightarrow X$ are said to be *path homotopic* (denoted $f \sim g$) if they are homotopic relative to the set $\{0, 1\}$. In other words, $f \sim g$ if there exists a homotopy (continuous map) $H : I \times I \rightarrow X$ such that

$$H(s, 0) = f(s) \quad \text{and} \quad H(s, 1) = g(s), \quad \forall s \in I,$$

$$H(0, t) = f(0) = g(0) \quad \text{and} \quad H(1, t) = f(1) = g(1), \quad \forall t \in I,$$

the second line meaning that f and g begin on the same point and end on the same point. In fact, for any $p, q \in X$, path homotopy gives an equivalence relation on the set of all paths from p to q . We call the equivalence class (denoted $[f]$) of a path f its *path class*.

Suppose f, g are two paths such that $f(1) = g(0)$. We define their composition to be the path

$$(f \circ g)(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2, \\ g(2s - 1) & 1/2 \leq s \leq 1. \end{cases}$$

A *loop* in X based at a point $p \in X$ is a path $f : I \rightarrow X$ such that $f(0) = f(1) = p$.

Definition: The *fundamental group* of X based at p is the set of path classes of loops based at p with the group operation of composition as described above. We will denote it by $\pi_1(X, p)$. If X is path connected, then the fundamental groups of X based at distinct points $p \neq q$ are isomorphic.

3 Introduction to Ricci Flow

Ricci flow was first introduced in Richard Hamilton's foundational paper, *Three-manifolds with positive Ricci curvature* (1982). The idea is that Ricci flow tends to smooth out uneven distributions by shaping the metric of a manifold. Objects become more "round", and the aim is for regions of a manifold to decompose into one of the eight Thurston geometries.

3.1 Ricci Curvature

Informally speaking, the Riemann curvature tensor measures the variation between the geometry given by the Riemannian metric and the geometry of standard Euclidean space. On the other hand, Ricci curvature takes the trace and measures the deviation between the volume of a ball in a Riemannian manifold versus one in Euclidean space.

Definition: The Ricci tensor identifies the tangent plane $T_p\mathcal{M}$ (for some $p \in \mathcal{M}$) with a symmetric bilinear map. Let $\{e_i\}$ be an orthonormal basis of $T_p\mathcal{M}$ (using Einstein's summation convention to simplify notation). From the Riemann curvature tensor, we take the trace of the linear map $x \mapsto \text{Rm}(x, y)z$ to get

$$\text{Ric}(y, z) = \sum_i \langle \text{Rm}(e_i, y)z, e_i \rangle.$$

In the case that $y = z$, given that we pick an orthonormal basis such that $e_1 = y$, the Ricci curvature is the sum of the sectional curvatures of planes that pass through e_1 and a different vector e_i (where $i \neq 1$).

We will now introduce the Lie derivative, which intuitively measures how an object (a function, a differential form, or in general a tensor) changes as it passes through some vector field. This generalizes the idea of a directional derivative, which is only well defined in Euclidean space.

Definition: Suppose X and W are smooth vector fields on \mathcal{M} and $\varphi(t) : \mathcal{M} \rightarrow \mathcal{M}$ is a one-parameter family of diffeomorphisms such that $\varphi(0)$ is the identity map and

$$\frac{d}{dt}[\varphi(t)] = X.$$

Then for any $p \in \mathcal{M}$, we may define a vector $(\mathcal{L}_X W)_p$ called the *Lie derivative* of W with respect to X at p , by

$$(\mathcal{L}_X W)_p = \left. \frac{d}{dt} \right|_{t=0} (\varphi_{-t}^* W)_{\varphi_t(p)},$$

where we pull W back to the point p using the flow φ of X and then differentiate.

In the case of two vector fields, say Y and Z , the Lie derivative of the metric g with respect to a vector field X is defined as

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).$$

If we choose $Y = \frac{\partial}{\partial x^i}$ and $Z = \frac{\partial}{\partial x^j}$, where the components of the metric are written as $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, then the Lie derivative can be written as

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

Furthermore, if $X = \nabla f$ is a gradient vector field, then the Lie derivative simplifies to

$$(\mathcal{L}_{\nabla f} g) = \nabla(\nabla f) + \nabla(\nabla f) = 2\nabla^2 f. \quad (3.1)$$

3.2 Ricci Solitons

As Hamilton had stated in 1982 in his foundational paper about Ricci flow, the process is governed by a partial differential equation (PDE) that evolves the metric tensor:

$$\frac{\partial}{\partial t}(g(t)) = -2\text{Ric}_g \quad (3.2)$$

with $g(0) = g_0$, where g_0 is the initial metric. From PDEs, we see that this model is very similar to the equation

$$\frac{\partial u}{\partial t} = k\Delta u$$

for heat and diffusion (where k is a proportionality constant). Because of the negative sign in equation (3.2), we see that the metric solution shrinks in the direction of positive Ricci curvature, and expands in the direction of negative Ricci curvature.

Definition: A solution $g(t)$ of the Ricci flow equation on \mathcal{M}^n is called a *Ricci soliton* if there exists a positive function $\sigma(t)$ (essentially a scaling factor) and a family $\varphi(t) : \mathcal{M}^n \rightarrow \mathcal{M}^n$ of diffeomorphisms (time dependent) such that

$$g(t) = \sigma(t)\varphi(t)^*g(0). \quad (3.3)$$

Ricci solitons can be thought of as "fixed points" of the Ricci flow, taking into account internal symmetries and rescalings. In other words, we can imagine a manifold with an evolution function in the quotient space of Riemannian metrics modulo diffeomorphisms and rescalings. We will derive the general Ricci soliton equation as follows. First, we differentiate equation (3.3) to get

$$\frac{\partial}{\partial t}(g(t)) = \frac{\partial}{\partial t}[\sigma(t)]\varphi(t)^*g(0) + \sigma(t) \cdot \frac{\partial}{\partial t}[\varphi(t)^*g(0)].$$

Using the definition of the Lie derivative with $X = \frac{d}{dt}[\varphi(t)]$, substituting the Ricci flow equation, and setting $t = 0$, we obtain

$$-2\text{Ric}_g = \sigma'(0)g_0 + \sigma(0)(\mathcal{L}_X g_0). \quad (3.4)$$

Now we can substitute $\sigma'(0) = -\lambda$ and drop the subscripts. Through rescaling the Ricci curvature and applying $\varphi(0)$ as the identity diffeomorphism, it follows that

$$\text{Ric}_g + (\mathcal{L}_X g) = \lambda g. \quad (3.5)$$

This gives one form of the general Ricci soliton equation. We say that g is shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. These names suggest exactly what we expect from a geometric point of view. If $\lambda > 0$, then over time the metric approaches zero. If $\lambda = 0$, then we obtain a limiting solution, as in the cigar soliton. Finally, if $\lambda < 0$, the metric expands without bound.

3.3 Gradient Ricci Solitons

Substituting equation (3.1) into equation (3.4) without rescaling constants, we arrive at the defining equation for a gradient Ricci soliton.

Definition: A *gradient Ricci soliton* (GRS) is a quadruple $(\mathcal{M}^n, g, f, \lambda)$ that satisfies the condition

$$\text{Ric}_g + \nabla^2 f = \frac{\lambda}{2}g,$$

with the same classification as above for $\lambda = 1$ (shrinking), $\lambda = 0$ (steady), and $\lambda = -1$ (expanding). Note that GRS are only defined for Ricci solitons with the additional assumption that $X = \nabla f$ is a gradient vector field.

3.4 Simple Examples of Solitons

One notable example of a steady Ricci soliton in dimension $n = 2$, due to Hamilton, is called the *cigar soliton* Σ , where we examine \mathbb{R}^2 with the metric

$$g_\Sigma = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2},$$

which satisfies the Ricci flow equation (3.2). With the 1-parameter group of diffeomorphisms $\phi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\phi_t(x, y) = (e^{-2t}x, e^{-2t}y),$$

g_Σ satisfies the Ricci soliton equation (3.3), so it is an example of the steady case. The cigar soliton is so named because it limits to a cylinder.

The easiest example to visualize is when $\text{Ric}_g = \frac{\lambda}{2}g$ and $f = 0$, such that we arrive at a shrinking GRS called the round sphere.

On \mathbb{R}^n (for $n \geq 3$), there are higher dimensional examples of steady GRS called Bryant solitons, which are symmetric with respect to rotations. Unfortunately these are more difficult to visualize.

4 Recent Work:

In his recent paper, William Wylie proved that if a complete Riemannian manifold \mathcal{M} induces a vector field such that the sum of the Ricci tensor and the Lie derivative of the

metric on the vector field is bounded from below by some positive multiple of the metric, then the fundamental group of \mathcal{M} is finite. The equation that he focused on is

$$\text{Ric}_g + \mathcal{L}_X g \geq \lambda g \quad (4.1)$$

for some $\lambda > 0$, which is one case of equation (3.5). As we recall, this corresponds to the shrinking Ricci soliton. García-Río and Fernández-López have already proved this statement under the additional assumption that $\|X\|$ is bounded by showing that \mathcal{M} is compact, and hence has a finite fundamental group, but Wylie was able to generalize their argument to the noncompact case.

4.1 Theorem 2 (Wylie)

If \mathcal{M} is a complete Riemannian manifold that satisfies equation (4.1), then the fundamental group of \mathcal{M} is finite.

4.2 Proofs of Intermediate Results

From this point on, assume that (\mathcal{M}, g) is a complete Riemannian manifold.

Definition Let $p \in \mathcal{M}$, and let $B_r(c)$ denote the open ball of radius r centered at a point c . We will define

$$H_p = \max\{0, \sup\{\text{Ric}_y(v, v)\}\},$$

where the supremum is taken over all vectors y and v such that $y \in B_1(p)$ and $\|v\| = 1$. Taking the maximum ensures that $H_p \geq 0$.

Lemma 3: Let $p, q \in \mathcal{M}$ such that $r = d(p, q) > 1$, and let γ be the minimal geodesic from p to q parametrized by arclength. Then

$$\int_0^r \text{Ric}(\gamma'(s), \gamma'(s)) ds \leq 2(n-1) + H_p + H_q.$$

Proof (of Lemma 3). Let ϕ be a piecewise smooth function such that $\phi(0) = \phi(r) = 0$. Let $\gamma : [0, r] \rightarrow \mathcal{M}^n$ be a geodesic parametrized by arclength such that $K > 0$ and $\text{Ric} \leq (n-1)K$. If $\{E_i\}_{i=1}^{n-1}$ is a parallel orthonormal frame along γ (and orthonormal to $T = \gamma'$), then the second variation of arclength equation (2.1) reduces to

$$0 \leq \int_0^r \left(\left| (\nabla_{\gamma'}(\phi E_i))^\perp \right|^2 - \langle \text{Rm}(\phi E_i, \gamma')\gamma', \phi E_i \rangle \right) ds$$

for $1 \leq i \leq n-1$. Summing over all such i gives us

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n-1} \int_0^r \left(\left| (\nabla_{\gamma'}(\phi E_i))^\perp \right|^2 - \langle \text{Rm}(\phi E_i, \gamma')\gamma', \phi E_i \rangle \right) ds \\ &\leq \int_0^r \left[\sum_{i=1}^{n-1} \left| \frac{\partial \phi}{\partial s} \right|^2 - \left(\phi^2 \sum_{i=1}^{n-1} \langle \text{Rm}(E_i, \gamma')\gamma', E_i \rangle \right) \right] ds \end{aligned}$$

$$\leq \int_0^r ((n-1)(\phi'(s))^2 - \phi^2(s)\text{Ric}(\gamma'(s), \gamma'(s)))ds. \quad (4.2)$$

In particular, let

$$\phi(s) = \begin{cases} s & 0 \leq s \leq 1, \\ 1 & 1 \leq s \leq r-1, \\ r-s & r-1 \leq s \leq r. \end{cases}$$

Let us add $\int_0^r \text{Ric}(\gamma'(s), \gamma'(s))ds$ to both sides of equation (4.3), obtaining

$$\begin{aligned} \int_0^r \text{Ric}(\gamma'(s), \gamma'(s))ds &\leq \int_0^r (n-1)(\phi'(s))^2 ds \\ &\quad + \int_0^r (1 - \phi^2(s))\text{Ric}(\gamma'(s), \gamma'(s))ds \end{aligned} \quad (4.3)$$

Using the fact that

$$\phi'(s) = \begin{cases} 1 & 0 \leq s \leq 1, \\ 0 & 1 \leq s \leq r-1, \\ -1 & r-1 \leq s \leq r, \end{cases}$$

we simplify the first term in the right-hand side of equation (4.3) to see that

$$\int_0^r (n-1)(\phi'(s))^2 ds = \int_0^1 (n-1)ds + \int_{r-1}^r (n-1)ds = 2(n-1). \quad (4.4)$$

Likewise, since $\phi(s) = 1$ in the interval $1 \leq s \leq r-1$, the second term in the right-hand side of equation (4.3) simplifies to yield

$$\begin{aligned} \int_0^r (1 - \phi^2(s))\text{Ric}(\gamma'(s), \gamma'(s))ds &= \int_0^1 (1 - \phi^2(s))\text{Ric}(\gamma'(s), \gamma'(s))ds \\ &\quad + \int_{r-1}^r (1 - \phi^2(s))\text{Ric}(\gamma'(s), \gamma'(s))ds \end{aligned} \quad (4.5)$$

In the interval $0 \leq s \leq 1$, we have that $0 \leq \phi(s) \leq 1$ and by definition $\text{Ric}(\gamma'(s), \gamma'(s)) \leq H_p$, so from the right-hand side of equation (4.5) it follows that

$$\int_0^1 (1 - \phi^2(s))\text{Ric}(\gamma'(s), \gamma'(s))ds \leq \int_0^1 (1 \cdot H_p)ds = H_p.$$

Similarly, in the interval $r-1 \leq s \leq r$, we have that $0 \leq \phi(s) \leq 1$ and $\text{Ric}(\gamma'(s), \gamma'(s)) \leq H_q$, so from the right-hand side of equation (4.5) it follows that

$$\int_{r-1}^r (1 - \phi^2(s))\text{Ric}(\gamma'(s), \gamma'(s))ds \leq \int_{r-1}^r (1 \cdot H_q)ds = H_q.$$

Piecing together these two inequalities together with equations (4.3), (4.4), and (4.5) yields

$$\int_0^r \text{Ric}(\gamma'(s), \gamma'(s))ds \leq 2(n-1) + H_p + H_q. \quad \square$$

Now our goal is to derive an upper bound on the distance $d(p, q)$ between p and q that depends only on $\|X\|$, H_p , and H_q .

Theorem 4: For any $p, q \in \mathcal{M}$,

$$d(p, q) \leq \max \left\{ 1, \frac{1}{\lambda} \left(2(n-1) + H_p + H_q + 2\|X_p\| + 2\|X_q\| \right) \right\}.$$

Proof of Theorem 4. Let us assume as before that $d(p, q) > 1$ and γ is the minimal geodesic from p to q . From equation (4.1), we see that

$$\begin{aligned} \int_0^r \text{Ric}(\gamma'(s), \gamma'(s)) ds &\geq \int_0^r [\lambda g(\gamma'(s), \gamma'(s)) - \mathcal{L}_X g(\gamma'(s), \gamma'(s))] ds \\ &\geq \lambda \int_0^r \|\gamma'(s)\| ds - \int_0^r [g(\nabla_{\gamma'(s)} X, \gamma'(s)) + g(\gamma'(s), \nabla_{\gamma'(s)} X)] ds \\ &\geq \lambda d(p, q) - \int_0^r \left[2 \frac{d}{ds} g(X, \gamma'(s)) \right] ds \\ &\geq \lambda d(p, q) - 2 \left[g(X, \gamma'(s)) \right]_0^r \\ &\geq \lambda d(p, q) - 2g_q(X, \gamma'(r)) + 2g_p(X, \gamma'(0)). \end{aligned} \quad (4.6)$$

Using the fact that $-g_p(X, \gamma'(0)) \leq \|X_p\|$ and $g_q(X, \gamma'(r)) \leq \|X_q\|$, equation (4.6) becomes

$$\int_0^r \text{Ric}(\gamma'(s), \gamma'(s)) ds \geq \lambda d(p, q) - 2\|X_p\| - 2\|X_q\|.$$

Recall from Lemma 3 that

$$\int_0^r \text{Ric}(\gamma'(s), \gamma'(s)) ds \leq 2(n-1) + H_p + H_q,$$

so

$$2(n-1) + H_p + H_q \geq \lambda d(p, q) - 2\|X_p\| - 2\|X_q\|.$$

We solve for $d(p, q)$ to get

$$1 < d(p, q) \leq \frac{1}{\lambda} \left(2(n-1) + H_p + H_q + 2\|X_p\| + 2\|X_q\| \right),$$

which proves Theorem 4. □

4.3 Proof of Theorem 2

Proof. Let $\phi : \widetilde{\mathcal{M}} \rightarrow \mathcal{M}$ be the covering map, where $\widetilde{\mathcal{M}}$ is the universal covering of \mathcal{M} . Then equation (4.1) holds for $\widetilde{\mathcal{M}}$ under the pullback metric ϕ^*g and pullback vector field, \widetilde{X} . Fix $\tilde{p} \in \widetilde{\mathcal{M}}$, write $\phi(\tilde{p}) = p$, and let $h \in \pi_1(\mathcal{M})$ correspond to a deck transformation on $\widetilde{\mathcal{M}}$.

Since the open balls $B_1(\tilde{p})$ and $B_1(h(\tilde{p}))$ are isometric,

$$H_{\tilde{p}} = H_{h(\tilde{p})} \quad \text{and} \quad \|\widetilde{X}_{\tilde{p}}\| = \|\widetilde{X}_{h(\tilde{p})}\|.$$

By the definition of ϕ , there exists a neighborhood $B_r(p)$ (without loss of generality, we can assume that the neighborhood is an open ball of radius $r > 0$) such that every component of $\phi^{-1}(B_r(p))$ is mapped homeomorphically onto $B_r(p)$ via ϕ . Taking the distance between \tilde{p} and $h(\tilde{p})$ and applying Theorem 4 gives us the upper bound

$$\begin{aligned} d(\tilde{p}, h(\tilde{p})) &\leq \max \left\{ 1, \frac{1}{\lambda} \left(2(n-1) + H_{\tilde{p}} + H_{h(\tilde{p})} + 2\|X_{\tilde{p}}\| + 2\|X_{h(\tilde{p})}\| \right) \right\} \\ &= \max \left\{ 1, \frac{2}{\lambda} \left((n-1) + H_{\tilde{p}} + 2\|X_{\tilde{p}}\| \right) \right\} = N \end{aligned} \quad (4.7)$$

for any $h \in \pi_1(\mathcal{M})$. N is independent of h , so if $\pi_1(\mathcal{M})$ were infinite, then there could only be finitely many $h \in \pi_1(\mathcal{M})$ such that $B_r(h(\tilde{p})) \in B_{(N+2r+1)}(\tilde{p})$, because of volume considerations (in other words, for a given $r > 0$, we can only fit finitely many non-overlapping balls of radius r into a ball of finite radius $N + 2r + 1$).

So there must exist $h' \in \pi_1(\mathcal{M})$ such that $d(\tilde{p}, h'(\tilde{p})) \geq N + 2r + 1$; let us choose h' such that $d(h(\tilde{p}), h'(\tilde{p})) = 2r$. From the triangle inequality, we have that

$$d(\tilde{p}, h'(\tilde{p})) \leq d(\tilde{p}, h(\tilde{p})) + d(h(\tilde{p}), h'(\tilde{p})),$$

so it follows that

$$\begin{aligned} d(\tilde{p}, h(\tilde{p})) &\geq d(\tilde{p}, h'(\tilde{p})) - d(h(\tilde{p}), h'(\tilde{p})) \\ &\geq N + 2r + 1 - 2r \\ &= N + 1, \end{aligned}$$

which is a contradiction of equation (4.7). Hence, the fundamental group of \mathcal{M} is finite, which proves Theorem 2. \square

5 Conclusion

We hope that this paper provides some geometric insight into the study of Ricci solitons. As a method begun by Richard Hamilton, Ricci flow continues to lead to areas of active research in geometry, analysis, topology, and other related fields.

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