

# PARADOXICAL DECOMPOSITION AND AMENABILITY

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ABSTRACT. In this article, we will first study various paradoxes such as the famous Banach-Tarski paradox, and discuss in detail the techniques to construct these paradoxes. We point out that these paradoxes preclude the existence of a certain measure. In section two we introduce the notion of amenability, the existence of a certain measure on a group. We give examples of groups that bear this kind of measure, and prove some basic properties of them. As an application of the theory of amenable groups, we show that Lebesgue measure on  $\mathbb{R}$  and  $\mathbb{R}^2$  has an isometry-invariant, finitely additive extension to all sets. Finally, we introduce the concept of supramenability, the existence of a certain measure normalizing an arbitrary subset, and point out its connection with growth rate of groups. Using theory of growth rate, we show that abelian groups are not only amenable but supramenable and no paradoxical subsets exist in  $\mathbb{R}$ .

## 1. PARADOXICAL DECOMPOSITION

In *Dialogues Concerning Two New Sciences*, Galileo observed that there is a one-to-one correspondence between the set of positive integers and the set of squares: “If I inquire how many roots there are, it can not be denied that there are as many as there are numbers because every number is the root of some square.” This seems paradoxical because the set of all positive integers, containing the set of nonsquares, seems more numerous than the set of squares. Galileo deduced from this that “the attributes of “equal”, “greater”, and “less”, are not applicable to infinite, but only to finite, quantities”. This was later clarified by Cantor’s theory of cardinality. Let  $\mathbb{N}$  denote the set of natural numbers,  $A$  the set of squares, and  $B$  the set of nonsquares.  $X \sim Y$  means there is a bijection between  $X$  and  $Y$ . What Galileo observed is that  $\mathbb{N} \sim A$  but  $A \subsetneq \mathbb{N}$ . The operation  $\sim$  appears to preserve size but here it seems to change size. It is not hard to see that  $\mathbb{N} \sim B$ . We say  $\mathbb{N} \sim A \sim B$  is a paradoxical decomposition of  $\mathbb{N}$ . This idea of decomposition can be generalized with group action.

**Definition 1.1** (Equidecomposability). Let  $G = (G, \cdot)$  be a group acting on a space  $X$  and  $A, B, E \subset X$ .

- (i)  $A$  and  $B$  are finitely  $G$ -equidecomposable, written  $A \sim B$ , if there exist finite partitions  $A = \cup_{i=1}^n A_i$  and  $B = \cup_{i=1}^n B_i$  and group elements  $g_1, \dots, g_n \in G$  such that  $B_i = g_i A_i$  for all  $1 \leq i \leq n$ .

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- (ii)  $E$  is finitely  $G$ -paradoxical if  $E$  can be partitioned into two disjoint subsets  $A$  and  $B$  such that  $A \sim B \sim E$ .

Similarly, we may define countably equidecomposable and countably paradoxical by replacing finite with countable in the above definition. If not mentioned, we assume we are talking about finite equidecomposability paradoxes. It is easy to see that  $\sim$  is an equivalence relation: if  $A, B$  are equidecomposable with  $n$  pieces and  $B, C$  are equidecomposable with  $m$  pieces, then  $A \sim C$  with at most  $mn$  pieces. We mostly consider the possibility of paradoxes when  $X$  is a metric space and  $G$  is a subgroup of the group of isometries of  $X$ . In the case that  $G$  is the full group of isometries of  $X$ , a subset  $E \subset X$  is paradoxical means  $E$  is (finitely)  $G$ -paradoxical where  $G$  is the group of isometries of  $X$ .

We are interested in paradoxical sets not only because it is paradoxical, but also because it precludes the existence of finitely additive, invariant measure that measures all sets.

*Remark 1.2.* Let  $G$  acts on  $X$ . Suppose  $\mu$  is finitely (resp. countably) additive,  $G$ -invariant measure on  $\mathcal{P}(X)$ . Let  $E$  be  $G$ -paradoxical witnessed by  $A_i, g_i, B_j, h_j$ . Then  $\mu(E) \geq \sum \mu(A_i) + \sum \mu(B_j) = \sum \mu(g_i A_i) + \sum \mu(h_j B_j) \geq \mu(\bigcup g_i A_i) + \mu(\bigcup h_j B_j) = \mu(E) + \mu(E) = 2\mu(E)$ . It follows that any finitely (reps. countably) additive and  $G$ -invariant measure on  $\mathcal{P}(X)$  must give either a zero measure or an infinite measure to a finitely (resp. countably)  $G$ -paradoxical set. Therefore, paradoxical sets provide significant obstructions to constructing measures that measure all sets.

**Example 1.3.** If  $G$  acts transitively on  $X$ , then two finite subsets of  $X$  are finitely  $G$ -equidecomposable iff they have the same cardinality, and any two countably infinite subsets of  $X$  are countably  $G$ -equidecomposable. In particular, any countably infinite subset of  $X$  is countably  $G$ -paradoxical. The case when  $X = \mathbb{N}$  and  $G = S_{\mathbb{N}}$  is discussed in the first paragraph.

Before we move on to geometric paradoxes, we first need a powerful tool. Whenever one has an equivalence relation on the collection of subsets of a set, one may define another relation,  $\preceq$ , by  $A \preceq B$  iff  $A$  is equivalent to a subset of  $B$ . Then  $\preceq$  is a reflexive and transitive relation on the equivalence classes. If the cardinality relation is used, i.e.  $A \sim B$  if there is a bijection from  $A$  to  $B$ , then Schröder-Berstein theorem of classical set theory states that  $\preceq$  is antisymmetric as well, i.e.  $A \preceq B$  and  $B \preceq A$  implies  $A \sim B$ . Thus  $\preceq$  is a partial order on the equivalence class. Banach realized that the proof of the Schröder-Berstein theorem could be generalized to any equivalence relation satisfying two abstract properties and in particular the generalization applies to  $G$ -equidecomposability. From now on  $A \preceq B$  is only used in the context of equidecomposability:  $A \preceq B$  means  $A$  is finitely (or countably)  $G$ -equidecomposable with a subset of  $B$ .

**Theorem 1.4** (Banach-Schröder-Berstein theorem). *Let  $G$  act on  $X$  and  $A, B \subset X$ . If  $A \preceq B$  and  $B \preceq A$ , then  $A \sim B$ . Thus  $\preceq$  is a partial ordering of the  $\sim$ -classes in  $\mathcal{P}(X)$ .*

*Proof.* Observe that the relation  $\sim$  satisfies the following conditions:

- (a) if  $A \sim B$  then there is a bijection  $f : A \rightarrow B$  such that  $C \sim g(C)$  whenever  $C \subset A$ ;
- (b) if  $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$ , and if  $A_1 \sim B_1$  and  $A_2 \sim B_2$ , then  $A_1 \cup A_2 \sim B_1 \cup B_2$ .

Let  $f : A \rightarrow B_1$  and  $g : A_1 \rightarrow B$  be bijections given by the assumption, where  $A_1 \subset A$  and  $B_1 \subset B$ . Let  $C_0 = A \setminus A_1$  and define  $C_{n+1} = g^{-1}(f(C_n))$  by induction; let  $C = \bigcup_{n=0}^{\infty} C_n$ . It is easy to see  $g(A \setminus C) = B \setminus f(C)$ . So  $A \setminus C \sim B \setminus f(C)$  since  $g$  is a bijection. Also  $C \sim f(C)$  since  $f$  is bijection. It follows that  $A = C \cup (A \setminus C) \sim f(C) \cup (B \setminus f(C)) = B$ .  $\square$

Note that this proof serves as a proof of the classical Schröder-Berstein theorem as well, since the cardinality relation satisfies properties (a) and (b). An immediate corollary below dramatically eases the verification of equidecomposability and shows that every paradoxical set admits a paradoxical decomposition.

**Corollary 1.5.**  $E \subset X$  is  $G$ -paradoxical iff there are disjoint sets  $A, B \subset E$  with  $A \cup B = E$  and  $A \sim E \sim B$ .

*Proof.* Suppose  $A, B$  are disjoint subsets of  $E$  with  $A \sim E \sim B$ . Then  $E \sim B \subset E \setminus A \subset E$ . It follows from Banach-Schröder-Berstein theorem that  $E \setminus A \sim E$ . Hence  $A$  and  $E \setminus A$  give a paradoxical decomposition of  $E$ .  $\square$

Let us first establish a countable equidecomposability paradox on  $\mathbb{R}$ .

**Theorem 1.6** (Vitali 1905 AC<sup>1</sup>). *Let  $\mathbb{R}$  act on itself by translations. Then  $[0, 1]$  and  $\mathbb{R}$  are countably  $\mathbb{R}$ -equidecomposable.*

*Proof.* By the Banach-Schröder-Berstein theorem, it suffices to show that some subset of  $[0, 1]$  is countably  $\mathbb{R}$ -equidecomposable with  $\mathbb{R}$ . Consider the equivalence relation on  $\mathbb{R}$  given by  $x \sim y$  iff  $x - y \in \mathbb{Q}$  and let  $\mathbb{R}/\mathbb{Q}$  denote all the equivalence classes. By the axiom of choice, for each equivalence class, we may choose a representative for some number in  $[0, \frac{1}{2}]$ . Thus we can partition  $\mathbb{R} = \bigcup_{x \in E} x + \mathbb{Q}$  for some  $E \subset [0, \frac{1}{2}]$ . Since  $\mathbb{R} \curvearrowright \mathbb{R}$  is transitive, by example 1.3  $\mathbb{Q} \cap [0, \frac{1}{2}]$  is countably  $\mathbb{Q}$ -equidecomposable with  $\mathbb{Q}$ . Therefore,

$$\mathbb{R} = \bigcup_{x \in E} x + \mathbb{Q} \sim \bigcup_{x \in E} x + \left( \mathbb{Q} \cap \left[ 0, \frac{1}{2} \right] \right) \subset [0, 1].$$

$\square$

Of course, the same theorem holds if  $[0, 1]$  is replaced by any other interval. As a consequence of the Banach-Schröder-Berstein theorem, any subset of  $\mathbb{R}$  containing an interval is countably  $\mathbb{R}$ -equidecomposable with  $\mathbb{R}$ . In particular, any subset of  $\mathbb{R}$  that contains an interval is countably  $\mathbb{R}$ -paradoxical.

Following remark 1.2, any countably additive translation-invariant measure that measures every subset of  $\mathbb{R}$  must assign a zero or infinite measure to any set containing an interval. Moreover, observe that any countably additive translation-invariant measure on  $\mathcal{P}(\mathbb{R}^n)$  induces a countably additive translation-invariant measure on the subsets of  $\mathbb{R}$ , by the correspondence  $E \leftrightarrow E \times [0, 1]^{n-1}$ . Therefore,

**Corollary 1.7** (AC). There is no countably additive, translation invariant measure defined on all subsets of  $\mathbb{R}^n$  and normalizing  $[0, 1]^n$ .

This corollary naturally leads to the question of whether measures exists satisfying a weaker set of conditions. The usual compromise is to restrict the measure to Lebesgue measurable sets and live with the fact that some sets are not Lebesgue

<sup>1</sup>In this article, theorem whose proof uses the axiom of choice will be marked by AC.

measurable. Another obvious compromise is to require only finite additivity instead of countable additivity. Finitely additive measures had been studied prior to Lebesgue, and it is natural to ask whether there exists a finitely additive, isometry-invariant measure defined for all subsets of  $\mathbb{R}^n$ . So, next we turn our attention to finite geometric equidecomposability paradoxes on  $\mathbb{R}^2$ .

**Sierpiński-Mazurkiewicz paradox.**  $SO(2) \times \mathbb{R}^2 \subset \text{Isom}(\mathbb{R}^2) = O(2) \times \mathbb{R}^2$  is the subgroup of orientation-preserving isometries of the isometries of  $\mathbb{R}^2$ . Let  $SO(2) \times \mathbb{R}^2$  acts on  $\mathbb{R}^2$  via  $(\theta, v).x = e^{i\theta} + v$  for  $v \in \mathbb{R}^2$  and  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ , where  $e^{i\theta}$  denotes the counterclockwise rotation by  $\theta$  around the origin. Does there exist a  $SO(2) \times \mathbb{R}^2$ -paradoxical subset of  $\mathbb{R}^2$ ? It turns out the key is to study the paradoxical sets with respect to the left translation action of a group on itself. If the group  $G$  is paradoxical with respect to the left translation action, we say  $G$  is paradoxical.

Recall that a free group  $F$  with generating set  $S$  is the group of all reduced words with letter from  $\{s, s^{-1} : s \in S\}$ , where a word is reduced if there are no adjacent letters of the form  $ss^{-1}$  or  $s^{-1}s$ . The group operation is concatenation, and the identity is the empty word. For example, the free group on two generators  $\{a, b\}$  consists of words of the form  $a, a^2, ab, a^{-2}, bab, ba^{-1}b^{-1}a$ , etc., and all of these elements are considered distinct. We shall occasionally be interested in the action of a semigroup  $S$  on a set  $X$ , where a semigroup is a set with an associative binary operation and an identity. A free semigroup with a generating set  $T$  is simply the set of all words with letter from  $T$  and with concatenation as the semigroup operation.

Free groups are the paradigms of paradoxical groups.

**Proposition 1.8.** A free group of rank 2,  $F_2$ , is a paradoxical group.

*Proof.* Let  $a, b$  be the two generators of  $F_2$ . For each  $x \in \{a^{\pm 1}, b^{\pm 1}\}$ , let  $W(x)$  denote the set of reduced words that begin with  $x$ . We can partition  $F_2 = \{1\} \sqcup W(a) \sqcup W(b) \sqcup W(a^{-1}) \sqcup W(b^{-1})$ , where 1 denote the empty word. Furthermore, observe that  $aW(a^{-1}) = F_2 \setminus W(a)$ . This implies  $W(a) \cup W(a^{-1}) \sim F_2$ . Similarly,  $W(b) \cup W(b^{-1}) \sim F_2$ .  $\square$

The most important technique in constructing a paradoxical decomposition is the transfer of a replicating partition of a paradoxical group to a set on which it acts. A paradoxical decomposition of a group is easily lifted to a set on which the group acts freely, i.e. the action has no nontrivial fixed points.

**Proposition 1.9 (AC).** If  $G$  is paradoxical and  $G$  acts freely on  $X$ , then  $X$  is finitely  $G$ -paradoxical.

*Proof.* By the axiom of choice, we can partition  $X$  as  $X = \sqcup_{x \in E} G.x$  for some subset  $E$  of  $X$ . Let  $A, B \subset G$  witness that  $G$  is paradoxical, i.e.  $G = A \sqcup B$  with  $A \sim G \sim B$ . Observe that

$$X = \sqcup_{x \in E} G.x = \sqcup_{x \in E} (A.x \sqcup B.x) = (\sqcup_{x \in E} A.x) \sqcup (\sqcup_{x \in E} B.x)$$

with  $\sqcup_{x \in E} A.x \sim \sqcup_{x \in E} G.x = X$  and  $\sqcup_{x \in E} B.x \sim \sqcup_{x \in E} G.x = X$ . Hence  $X$  is finitely  $G$ -paradoxical.  $\square$

Since a subgroup of a group acts freely by left translation on the whole group, we have the following immediate corollary.

**Corollary 1.10** (AC). A group with a paradoxical subgroup is paradoxical. In particular, any group which contains an isomorphic copy of  $F_2$  is paradoxical.

There is an easy converse to Proposition 1.9.

**Proposition 1.11.** For arbitrary action  $G \curvearrowright X$ , if  $X$  is  $G$ -paradoxical then  $G$  is a paradoxical group.

*Proof.* Let  $x \in X$  be an arbitrary. Let  $A_i, B_j \subset X, g_i, h_j \in G$  witness that  $X$  is  $G$ -paradoxical, i.e.  $X = \cup_i g_i A_i = \cup_j h_j B_j$ . Let  $A = \cup_i A_i$ ,  $B = \cup_j B_j$ , and  $G_Y = \{g \in G : g.x \in Y\}$  for  $Y \subset X$ . It is not hard to see that  $G_{A_i}, G_{B_j}, g_i, h_j$  witness that  $G$  is paradoxical, and  $G_A \sim G \sim G_B$ .  $\square$

Sierpiński and Mazurkiewicz proved the existence of a paradoxical subset of plane using the idea in Proposition 1.9. However,  $SO(2) \times \mathbb{R}^2$  does not contain a free subgroup of rank 2 but instead a free semigroup of rank 2. The existence of a free subsemigroup of  $SO(2) \times \mathbb{R}^2$  is the key to prove the existence of a paradoxical subset of the plane. The semigroup ping-pong lemma below gives a sufficient condition when two elements of a group generates a free semigroup.

**Lemma 1.12** (Semigroup ping-pong lemma). *Let  $G$  acts on  $X$  and let  $g, h \in G$ . Suppose there exists nonempty subset  $A$  of  $X$  such that  $gA$  and  $hA$  are disjoint subsets of  $A$ . Then  $g, h$  generate a free semigroup.*

*Proof.* Let  $w, w'$  be two distinct words generated by  $g, h$ . If  $w$  is empty, then  $w'$  is nonempty so  $w'A \subsetneq gA$  or  $w'A \subsetneq hA$  whereas  $wA = A$ . Without loss of generality assume both  $w, w'$  are nonempty. Suppose  $w = st$  and  $w' = s't'$ , where  $s, s' \in \{g, h\}$ . If  $s \neq s'$ , then  $wA$  and  $w'A$  are disjoint; if  $s = s'$ , then  $t \neq t'$  and the lemma follows by induction on the length of words.  $\square$

**Theorem 1.13** (Sierpiński-Mazurkiewicz paradox 1914). *There exists a nonempty finitely  $SO(2) \times \mathbb{R}^2$ -paradoxical subset of the plane.*

*Proof.* Identify  $\mathbb{R}^2$  with  $\mathbb{C}$ . Let  $g$  to be the rotation  $g.x = \omega x$  for some transcendental phase  $\omega = e^{i\theta}$ . Such a phase must exist since the set of algebraic numbers on the unit circle is countable. Let  $h$  be the translation  $h.x = x + 1$ . Observe that  $g$  and  $h$  act on the set  $A$  of polynomials in  $\omega$  with nonnegative integer coefficients, and that  $gA$  and  $hA$  are disjoint. By the semigroup ping-pong lemma,  $g, h$  generates a free semigroup  $S$ . It is easy to see that  $S$  is countable, for  $S$  has the same cardinality as the set of finite binary strings. Note that each element in  $SO(2) \times \mathbb{R}^2$  has at most one fixed point. Let  $E$  be the set of fixed points of elements in  $S$ , which is at most countable. Let  $x \in \mathbb{R}^2 \setminus E$  and  $S.x$  be the  $S$ -orbit of  $x$ .  $g(S.x), h(S.x) \subset S.x$  and  $g(S.x) \cap h(S.x) = \emptyset$ . Therefore,  $S.x$  is finitely  $SO(2) \times \mathbb{R}^2$ -paradoxical.  $\square$

Note that the Sierpiński-Mazurkiewicz paradox does not invoke the axiom of choice. The proof of the Sierpiński-Mazurkiewicz paradox is quite constructive. Following the proof of Theorem 1.13, we see that the set  $S.(0, 0) = A = \{a_0 + a_1 e^{i\theta} + \dots + a_n e^{in\theta} : n \in \mathbb{N}, a_i \in \mathbb{N}\}$  is finitely  $SO(2) \times \mathbb{R}^2$ -paradoxical. Let  $g$  and  $h$  be as in the above proof.  $gA = \{a_1 e^{i\theta} + \dots + a_n e^{in\theta} : n \in \mathbb{N}, a_i \in \mathbb{N}\} \sim A$  and  $hA = \{a_0 + a_1 e^{i\theta} + \dots + a_n e^{in\theta} : n \in \mathbb{N}, a_i \in \mathbb{N}, a_0 \neq 0\} \sim A$ . The existence of such a set is by no means contradictory. After all,  $A$  is countable and so has measure zero; from measure theoretic point of view,  $A$  is paradoxical implies  $2 \cdot 0 = 0$ , which should not be surprising at all.

The ideas of the Sierpiński-Mazurkiewicz paradox form the foundation of much of the early history of geometric paradoxes. At the same year, Hausdorff discovered an amazing paradox on the sphere that led to the famous Banach-Tarski paradox.

**Hausdorff Paradox.** Hausdorff paradox shows that barring a countable subset the sphere is finitely  $SO(3)$ -paradoxical. The new feature in  $\mathbb{R}^3$  is that the group  $SO(3) \ltimes \mathbb{R}^3$  of rigid motions contains a free group of rank 2 (not just a free semigroup such as in  $SO(2) \ltimes \mathbb{R}^2$ ). The ping-pong lemma below gives a sufficient condition when two elements of a group generate a free semigroup.

**Lemma 1.14** (Ping-pong lemma). *Let  $G$  be a group acting on a set  $X$ , let  $\Gamma_1, \Gamma_2$  be two subgroups of  $G$ , and let  $\Gamma$  be the subgroup of  $G$  generated by  $\Gamma_1$  and  $\Gamma_2$ ; assume  $|\Gamma_1| \geq 3$  and  $|\Gamma_2| \geq 2$ . Assume there exist two nonempty subsets  $X_1, X_2$  in  $X$ , with  $X_2 \not\subset X_1$ , such that  $(\Gamma_1 \setminus \{1\})X_2 \subset X_1$  and  $(\Gamma_2 \setminus \{1\})X_1 \subset X_2$ . Then  $\Gamma \simeq \Gamma_1 * \Gamma_2$ , the free product of  $\Gamma_1$  and  $\Gamma_2$ .*

*Proof.* Let  $w$  be a nonempty reduced word with letters from disjoint union of  $\Gamma_1 \setminus \{1\}$  and  $\Gamma_2 \setminus \{1\}$ . We need to show that the element of  $\Gamma$  defined by  $w$  (again written  $w$ ) is not the identity.

If  $w = a_1 b_1 a_2 b_2 \cdots a_k$  with  $a_1, \dots, a_k \in \Gamma_1 \setminus \{1\}$  and  $b_1, \dots, b_{k-1} \in \Gamma_2 \setminus \{1\}$ , then

$$\begin{aligned} w(X_2) &= a_1 b_1 \cdots a_{k-1} b_{k-1} a_k(X_2) \subset a_1 b_1 \cdots a_{k-1} b_{k-1}(X_1) \\ &\subset a_1 b_1 \cdots a_{k-1}(X_2) \subset \cdots \subset a_1(X_2) \subset X_1. \end{aligned}$$

As  $X_2 \not\subset X_1$ , this implies  $w \neq 1$ .

If  $w = b_1 a_2 b_2 a_2 \cdots b_k$ , choose  $a \in \Gamma_1 \setminus \{1\}$ ; the previous argument shows that  $awa^{-1} \neq 1$  so that  $w \neq 1$ . If  $w = a_1 b_1 \cdots a_k b_k$ , choose  $a \in \Gamma_1 \setminus \{1, a_1^{-1}\}$  and again  $awa^{-1} \neq 1$  implies  $w \neq 1$ . If  $w = b_1 a_2 b_2 \cdots a_k$ , choose  $a \in \Gamma_1 \setminus \{1, a_k\}$  again  $awa^{-1} \neq 1$  implies  $w \neq 1$ .  $\square$

Hausdorff showed that if  $\phi$  and  $\rho$  are rotations through  $180^\circ$ ,  $120^\circ$ , respectively, about axes containing the origin, and if  $\cos 2\theta$  is transcendental where  $\theta$  is the angle between the axes, then  $\phi$  and  $\rho$  are free generators of  $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ . The approach we will be using in the lemma below is due to Świerczkowski. His approach avoids transcendental numbers, and uses perpendicular axes and the same angle for each rotation. It is worth mentioning that if  $SO(3) \times SO(3)$  is given the product topology, then  $\{(\phi, \rho) \in SO(3) \times SO(3) : \phi, \rho \text{ generate a free group of rank 2}\}$  is comeager and hence dense, but we shall be content by giving a specific pair of generators.

**Lemma 1.15.**  *$SO(3)$  has a free subgroup of rank 2.*

*Proof.* Let  $\phi$  and  $\rho$  be counterclockwise rotations around the  $z$ -axis and  $x$ -axis, each through the angle  $\arccos \frac{3}{5}$ . Then  $\phi^{\pm 1}, \rho^{\pm 1}$  are represented by matrices as follows:

$$\phi^{\pm 1} = \begin{bmatrix} \frac{3}{5} & \mp \frac{4}{5} & 0 \\ \pm \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho^{\pm 1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \mp \frac{4}{5} \\ 0 & \pm \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Let  $X_1 = 5^{\mathbb{Z}} \cdot \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{Z}, x \equiv \pm 3y \pmod{5}, z \equiv 0 \pmod{5} \right\}$  and let  $X_2 = 5^{\mathbb{Z}} \cdot \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{Z}, z \equiv \pm 3y \pmod{5}, x \equiv 0 \pmod{5} \right\}$ , where  $5^{\mathbb{Z}}$  denote the integral power of 5. Let  $X = X_1 \cup X_2$ . It is easy to check that  $X_2 \not\subset X_1$ ,  $(\Gamma_1 \setminus \{1\})X_2 \subset X_1$ , and  $(\Gamma_2 \setminus \{1\})X_1 \subset X_2$ , where  $\Gamma_1 = \langle \phi \rangle$  and  $\Gamma_2 = \langle \rho \rangle$ . It follows from the ping-pong lemma that  $\langle \phi, \rho \rangle = \langle \phi \rangle * \langle \rho \rangle$ , a free subgroup of rank 2.  $\square$

Each element of the free subgroup of  $SO(3)$  constructed above fixes all points on some line in  $\mathbb{R}^3$  so the action is not free and Proposition 1.9 can not be applied directly. A naive approach to this difficulty turns out to be fruitful.

**Theorem 1.16** (Hausdorff paradox 1914 AC). *There is a countable subset  $E$  of  $S^2$  such that  $S^2 \setminus E$  is finitely  $SO(3)$ -paradoxical.*

*Proof.* Let  $F_2 \subset SO(3)$  be the free subgroup constructed in the above lemma. Each nontrivial rotation in  $F_2$  has exactly two fixed points on the unit sphere, namely the intersection of the axis of rotation with  $S^2$ . Let  $E$  be the collection of all such fixed points; since  $F_2$  is countable, so is  $E$ . Now, if  $p \in S^2 \setminus E$  and  $g \in F_2$ , then  $g.p \in S^2 \setminus E$  as well: for if  $h$  fixes  $g.p$ , then  $g^{-1}hg$  fixes  $p$ . Hence  $F_2$  acts freely on  $S^2 \setminus E$ . By Proposition 1.9, we may transfer the replicating partition of  $F_2$  to  $S^2 \setminus E$  and so  $S^2 \setminus E$  is finitely  $SO(3)$ -paradoxical.  $\square$

A countable subset of the sphere can be dense, and so the paradoxical nature of Hausdorff paradox is not apparent yet. Still, countable sets are very small in measure compared to the whole sphere. We shall see immediately how the countable set  $E$  can be absorbed completely, yielding the Banach-Tarski paradox:  $S^2$  is finitely  $SO(3)$ -paradoxical.

### Banach-Tarski Paradox.

**Theorem 1.17** (Banach-Tarski paradox on the sphere AC).  *$S^2$  is finitely  $SO(3)$ -paradoxical.*

*Proof.* By the Hausdorff paradox, it suffices to show  $S^2$  is finitely  $SO(3)$ -equidecomposable with  $S^2 \setminus E$ , where  $E$  is a countable subset of  $S^2$ . Let  $l$  be a line through the origin that misses the countable set  $E$ . Let  $A$  be the set of angles  $\theta$  such that for some  $n > 0$  and some  $p \in E$ ,  $\rho(p) \in E$  where  $\rho$  is the rotation about  $l$  through angle  $n\theta$ . Then  $A$  is countable, so we may choose an angle  $\theta$  not in  $A$ ; let  $\rho$  be the corresponding rotation about  $l$ . Then  $\rho^n(E) \cap E = \emptyset$  if  $n > 0$ . It follows that for all  $0 \leq m < n$ ,  $\rho^m(E) \cap \rho^n(E) = \emptyset$ . Let  $C = \cup_{n \geq 0} \rho^n(E)$ . Then  $S^2 = C \cup (S^2 \setminus C) \sim \rho(C) \cup (S^2 \setminus C) = S^2 \setminus E$  as desired.  $\square$

The above proof may be called proof by absorption, since it shows how a troublesome set can be absorbed in a way that essentially renders it irrelevant. The Banach-Tarski paradox on the unit sphere can be easily extended to one on the unit ball with absorption of the origin.

**Corollary 1.18** (Banach-Tarski paradox AC). *The unit ball  $B$  in  $\mathbb{R}^3$  is finitely  $SO(3) \times \mathbb{R}^3$ -paradoxical and so is  $\mathbb{R}^3$  itself.*

*Proof.* The radial correspondence  $E \mapsto \{\alpha E : 0 < \alpha \leq 1\}$  gives us a paradoxical decomposition of  $B \setminus \{0\}$  from the paradoxical decomposition of  $S^2$ . Hence it suffices to show that  $B$  is finitely  $SO(3) \times \mathbb{R}^3$ -equidecomposable with  $B \setminus \{0\}$ , i.e. a point can be absorbed. Let  $p = (0, 0, \frac{1}{2})$  and let  $\rho$  be a rotation of infinite order about an axis through  $p$  but missing the origin. Then, just as in the proof above, the set  $C = \{\rho^n(0) : n \geq 0\}$  can be used to absorb 0:  $\rho(C) = C \setminus \{0\}$  so  $B = C \cup (B \setminus C) \sim \rho(C) \cup (B \setminus C) = B \setminus \{0\}$ . If, instead, the radial correspondence of  $S^2$  with all of  $\mathbb{R}^3 \setminus \{0\}$  is used one gets a paradoxical decomposition of  $\mathbb{R}^3 \setminus \{0\}$  using rotations. Then exactly as for the unit ball,  $\mathbb{R}^3 \setminus \{0\} \sim \mathbb{R}^3$ , i.e.  $\mathbb{R}^3$  is finitely  $SO(3) \times \mathbb{R}^3$  paradoxical.  $\square$

The Banach-Tarski paradox is very bizarre indeed. A ball, which has a definite volume, may be taken apart into finitely many pieces that may be rearranged via rotation to form 2, or even  $10^9$  balls, each identical to the original one! Rotation preserves volume, and this is why this result has come to be known as a paradox. A resolution is that there may not be a volume for the rotations to preserve; the pieces in the decomposition will have to be Lebesgue non-measurable.

The following result on nonexistence of measure is an immediate corollary of the Banach-Tarski paradox and Remark 1.2. Note that to prove the assertion about  $\mathbb{R}^n$  it suffices to consider  $\mathbb{R}^3$  because a measure in a higher dimension induces one in  $\mathbb{R}^3$  as described before Corollary 1.7.

**Corollary 1.19** (AC). There is no finitely additive, rotation-invariant probability measure on  $\mathcal{P}(S^2)$  and for  $n \geq 3$  there is no finitely additive, isometry-invariant measure on  $\mathcal{P}(\mathbb{R}^n)$  normalizing the cube.

There are properties that are preserved by finite equidecomposability in  $\mathbb{R}^3$ , though volume is not one of them. If  $A$  is bounded, then so is any set finitely equidecomposable with  $A$ ; the same is true if  $A$  has nonempty interior. Banach and Tarski showed that any two subsets of  $\mathbb{R}^3$ , each having these two properties, are equidecomposable. Thus they generalized their already surprising result so that it applies to solids of any shape. The following strong form of the Banach-Tarski paradox is informally known as the "pea and the sun paradox", i.e. a pea can be chopped up into finitely many pieces and reassembled to form a ball size of the sun!

**Theorem 1.20** (Banach-Tarski paradox AC). *If  $A$  and  $B$  are two bounded subsets of  $\mathbb{R}^3$  with nonempty interior, then  $A$  and  $B$  are finitely  $SO(3) \times \mathbb{R}^3$ -equidecomposable.*

*Proof.* It suffices to show  $A \preceq B$ , for then by symmetry and Banach-Schröder-Berstein theorem  $A \sim B$ . Let  $K$  and  $L$  be solid balls such that  $A \subset K$  and  $L \subset B$ . Let  $n$  be large enough such that  $K$  may be covered by  $n$  (overlapping) copies of  $L$ . Now, if  $S = \sqcup_{i=1}^n L$ , then duplicating  $L$  by Banach-Tarski paradox and using translations to move the copies around gives  $L \sim S$ . Therefore,  $A \subset K \preceq S \sim L \subset B$ , so  $A \preceq B$ .  $\square$

The Banach-Tarski paradox can be easily extended to higher dimensions. The idea is simple with induction. Let  $n \geq 3$ . Except for the north and south pole,  $S^n$  can be cut into layers of scaled  $S^{n-1}$ . Apply the paradoxical decomposition for  $S^{n-1}$  to each of the layer to obtain a paradoxical decomposition of  $S^n$  minus the poles. Then absorb the poles as in Corollary 1.18.



**Theorem 1.21** (Banach-Tarski paradox in  $\mathbb{R}^{n \geq 3}$  AC).  $S^{n-1}$  is finitely  $SO(n)$  paradoxical for  $n \geq 3$ . If  $n \geq 3$ , the unit ball in  $\mathbb{R}^n$  is  $SO(n) \times \mathbb{R}^n$ -paradoxical and so is  $\mathbb{R}^n$  itself.

*Proof.* The Banach-Tarski paradox in  $\mathbb{R}^3$  shows that the result is true for  $n = 3$ . We proceed from there by induction. Consider  $S^n$  in  $\mathbb{R}^{n+1}$ . Suppose  $A_i, B_j \subset S^{n-1}$  and  $\sigma_i, \tau_j \in SO(n)$  witness that  $S^{n-1}$  is finitely paradoxical. Define  $\tilde{A}_i, \tilde{B}_j$  to partition  $S^n \setminus \{(0, \dots, 0, \pm 1)\}$ , by putting  $(x_1, \dots, x_n, z)$  in  $\tilde{A}_i$  or  $\tilde{B}_j$  according to which of the  $A_i, B_j$  contains  $(x_1, \dots, x_n)/|(x_1, \dots, x_n)|$ . Extend  $\sigma_i, \tau_j$  to  $\tilde{\sigma}_i, \tilde{\tau}_j$  by fixing the new axis. In matrix form,

$$\tilde{\sigma}_i = \begin{bmatrix} & & & 0 \\ & & & \vdots \\ & \sigma_i & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then  $\tilde{A}_i, \tilde{B}_j, \tilde{\sigma}_i, \tilde{\tau}_j$  provide a paradoxical decomposition of  $S^n \setminus \{(0, \dots, 0, \pm 1)\}$ . But any two-dimensional rotation of infinite order, viewed as rotation the last two coordinates and fixed the first  $n - 1$  coordinates, can be used as usual (see proof of Corollary 1.18 to show that  $S^n \setminus \{(0, \dots, 0, \pm 1)\} \sim S^n$ ). Hence by induction  $S^n$  is finitely paradoxical. The rest of the theorem follows exactly as Corollary 1.18 follows from the Banach-Tarski paradox on  $S^2$ .  $\square$

This concludes our discussion on paradoxical decomposition and nonexistence of measure as a consequence. In the next section, we will explore the opposite, where existence of certain invariant measure precludes any paradoxes.

## 2. AMENABILITY

In the first section we saw that the main idea in the construction of a paradoxical decomposition of a set was to first get a decomposition in the group acting on the set, and then transfer to the set. An almost identical theme pervades the construction of invariant measures on a set  $X$  acted upon by a group  $G$ . If there is a finitely additive, left-invariant measure defined on all subsets of  $G$ , then this measure on  $G$  can be used to produce a finitely additive,  $G$ -invariant measure defined on all subsets of  $X$ . Such measures on  $X$  yield that  $X$  is not  $G$ -paradoxical.

It was von Neumann who realized that such a transference of measures was possible, and he began classifying the groups that admit measures of this sort. In this section we study properties of the class of groups equipped with a finitely additive invariant measure. As an application, we obtain the nonexistence of Banach-Tarski-type paradoxes in  $\mathbb{R}$  and  $\mathbb{R}^2$ .

**Definition 2.1** (Amenability). A (discrete) group  $G$  is amenable if  $G$  admits a finitely additive, left-invariant probability measure on  $\mathcal{P}(G)$ .

*Remark 2.2* (Topological Amenability). We may generalize the notion of amenability to topological groups. Let  $G$  be a locally compact Hausdorff group.  $G$  is called (topologically) amenable if there exists a finitely additive, left-invariant probability measure on the Borel subsets of the group. Saying that  $G$  is amenable with the discrete topology (which is locally compact) is the same as saying  $G$  is amenable in the nontopological sense as in definition 2.1 since every set is Borel in the discrete topology. Note that a topological group that is amenable as an abstract group is

topologically amenable: simply restrict the measure on  $\mathcal{P}(G)$  to the Borel sets. If  $G$  is compact, then the Haar measure is a countably additive, left-invariant Borel probability measure; hence compact groups are topologically amenable. As an example,  $SO(3)$  is a compact subgroup of  $GL_3(\mathbb{R})$  so it is topologically amenable; however, we have shown  $SO(3)$  contains a free group of rank 2 in Lemma 1.15 and so it is a paradoxical group and hence not amenable as a discrete group.

Here is the observation made by von Neumann that a finitely additive invariant measure on a group  $G$  can be transferred to a measure of this sort on the set  $G$  acts on.

**Proposition 2.3.** Let  $G$  be an amenable group acting on  $X$ . Then there exists a finitely additive,  $G$ -invariant probability measure on  $\mathcal{P}(X)$ ; hence  $X$  is not finitely  $G$ -paradoxical.

*Proof.* Let  $x \in X$ . If  $\mu$  is a measure on  $G$ , define  $\nu : \mathcal{P}(X) \rightarrow [0, 1]$  by  $\nu(A) = \mu(\{g \in G : g.x \in A\})$ . It is easy to see that  $\nu$  is a finitely additive,  $G$ -invariant measure on  $\mathcal{P}(X)$  and  $\nu(X) = 1$ . Therefore,  $X$  is not  $G$ -paradoxical.  $\square$

A mean on  $G$  is a linear functional  $m : l^\infty(G) \rightarrow \mathbb{C}$  such that  $m$  is positive, i.e.  $f \geq 0 \implies m(f) \geq 0$ , and  $m$  is unital, i.e.  $m(\mathbf{1}) = 1$ . A mean is automatically bounded with norm one, for if  $f \in l^\infty(G)$  then there exists a  $\alpha \in \mathbb{T}$  such that

$$\begin{aligned} |m(f)| &= \alpha m(f) = m(\alpha f) = \operatorname{Re} m(\alpha f) \\ &= m(\operatorname{Re} \alpha f) \leq \|\operatorname{Re} \alpha f\|_\infty \leq \|\alpha f\|_\infty \leq \|f\|_\infty. \end{aligned}$$

So  $\|m\| = 1$  for every mean on  $G$ . The mean  $m$  is left-invariant if  $m(g.f) = m(f)$  for all  $g \in G$  and  $f \in l^\infty(G)$ , where  $g.f(h) = f(g^{-1}h)$ .

There is a canonical correspondence between means on  $G$  and finitely additive probability measure on  $\mathcal{P}(G)$ . Given each mean  $m$ , define a finitely additive probability measure  $\mu$  by  $\mu(A) = m(\chi_A)$ , where  $\chi_A$  is the characteristic function of  $A$ ; given a finitely additive measure  $\mu$ , define a mean  $m$  on  $G$  by  $m(f) = \int f d\mu$ . Thus, we have the following Proposition:

**Proposition 2.4.**  $G$  is amenable iff  $G$  admits a left-invariant mean.

Similar to remark 1.2, a measure on  $G$  precludes  $G$  from being paradoxical: if  $\mu$  is such measure and  $G$  is paradoxical, fix disjoint sets  $A, B \subset G$  such that  $A \sim G \sim B$  and note that  $\mu(G) \geq \mu(A \cup B) = \mu(A) + \mu(B) = 2\mu(G)$ , contradicting  $\mu(G) = 1$ . In other words, if  $G$  is paradoxical, then  $G$  is not amenable. In fact, a deep theorem of Tarski proved that the converse is true. See Wagon [2] for details.

**Theorem 2.5** (Tarski's theorem 1929 AC).  $G$  is not amenable iff  $G$  is paradoxical.

Next we give some classes of examples of amenable groups and summarize some closure properties of amenable groups.

**Proposition 2.6.**

- (i) Every finite group is amenable.
- (ii) (AC) If  $G$  is amenable and  $\Gamma \leq G$ , then  $\Gamma$  is amenable.
- (iii) If  $N \triangleleft G$ , then  $G$  is amenable  $\iff N, G/N$  are amenable.
- (iv) If  $G$  and  $H$  are amenable then so is  $G \times H$ .
- (v) (AC) If  $G$  is the direct union of a directed system of amenable groups  $\{G_\alpha : \alpha \in I\}$ , then  $G$  is amenable.

- (vi) (AC)  $\mathbb{Z}$  is amenable.
- (vii) (AC) Every abelian group is amenable.
- (viii) (AC) Every solvable group is amenable. In particular, every nilpotent group is amenable.

*Proof.*

- (i) Suppose  $G$  is finite. Then  $\mu(A) = |A|/|G|$  yields the desired measure on  $G$ .
- (ii) Let  $\mu$  be a measure on  $G$  and let  $\Gamma$  be a subgroup of  $G$ . Let  $R$  be a set of representatives for the collection of right cosets of  $\Gamma$  in  $G$ . Note that  $\nu(A) = \mu(AR) = \mu(\cup_{r \in R} Ar)$  is the desired measure on  $\Gamma$ .
- (iii) Let  $\mu$  be a measure on  $G$ . Define  $\nu : \mathcal{P}(G/N) \rightarrow [0, 1]$  by  $\nu(A) = \mu(\cup A)$ . It is easy to check that  $\nu$  is a measure on  $G/N$ . Conversely, let  $\mu, \nu$  be measures on  $N$  and  $G/N$ , respectively. Then  $\mu$  induces a measure  $\mu_C$  on each  $C \in G/N$  given by  $\mu_C(A) = \mu(g^{-1}A)$ , where  $\gamma \in C$ . As  $\mu$  is left-invariant,  $\mu_C$  is independent of the choice of  $g$ . Now observe that

$$\lambda(A) = \int_{G/N} \mu_C(A \cap C) d\nu(C)$$

defines a measure on  $G$ .

- (iv) Let  $\iota$  be the embedding  $h \mapsto (e_G, h)$  into  $G \times H$ . Then  $\iota(H) \triangleleft G \times H$  with  $G \times H / \iota(H) \simeq G$ . So by part (iii),  $G \times H$  is amenable if  $G$  and  $H$  are amenable.
- (v) Given  $G = \cup \{G_\alpha : \alpha \in I\}$  where each  $G_\alpha$  is amenable with measure  $\mu_\alpha$ , and for each  $\alpha, \beta \in I$  there exists  $\gamma \in I$  such that  $G_\alpha$  and  $G_\beta$  are each subgroups of  $G_\gamma$ . Consider the topological space  $[0, 1]^{\mathcal{P}(G)}$ . This space is compact by Tychonoff's theorem. For each  $\alpha \in I$ , let  $\mathcal{M}_\alpha$  consist of finitely additive probability measure  $\mu : \mathcal{P}(G) \rightarrow [0, 1]$  such that  $\mu(gA) = \mu(A)$  whenever  $g \in G_\alpha$ . Then each  $\mathcal{M}_\alpha \neq \emptyset$ , as witnessed by  $\mu(A) = \mu_\alpha(A \cap G_\alpha)$ . And it is easy to check  $\mathcal{M}_\alpha$  is a closed subset of  $[0, 1]^{\mathcal{P}(G)}$ . Since  $\mathcal{M}_\alpha \cap \mathcal{M}_\beta \supset \mathcal{M}_\gamma$  if  $G_\alpha, G_\beta \subset G_\gamma$ , the collection  $\{\mathcal{M}_\alpha : \alpha \in I\}$  has the finite intersection property. By compactness, there exists  $\mu \in \cap_{\alpha \in I} \mathcal{M}_\alpha$ , and such a  $\mu$  witnesses the amenability of  $G$ .
- (vi) Let  $\omega \in \beta\mathbb{N}$  be a free ultrafilter. Then

$$\mu(A) = \lim_{n \rightarrow \omega} \frac{|A \cap \{-n, \dots, n\}|}{2n + 1}$$

defines a measure on  $\mathbb{Z}$ . Here  $\lim_{n \rightarrow \omega} a_n$  denote the unique real number  $a$  such that for each neighborhood  $U$  of  $a$  the set  $\{n : a_n \in U\} \in \omega$ .

- (vii) Any group is the direct union of its finitely generated subgroups so by (v) a group is amenable iff all its finitely generated subgroups are amenable. Let  $G$  be a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups,  $G \simeq \mathbb{Z}^r \times \mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_m\mathbb{Z}$  for some  $r \geq 0$  and integers  $k_1 \mid k_2 \mid \dots \mid k_m$ . Hence any finitely generated abelian group is amenable by (iv), (i), and (vi).
- (viii) Let  $G$  be a solvable group. Then there exists normal series  $\{1\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$  such that  $G_{i+1}/G_i$  is abelian for all  $i < n$ . Therefore,  $G$  is amenable by (iii) with induction. Every nilpotent group is solvable so nilpotent groups are amenable.

□

**Example 2.7.** In  $\mathbb{R}$ ,  $\text{Isom}(\mathbb{R}) = O(1) \times \mathbb{R}$ . Since  $O(1) \times \mathbb{R}/\mathbb{R} \simeq O(1) \simeq \mathbb{Z}/2\mathbb{Z}$ , the normal series  $\{1\} \triangleleft \mathbb{R} \triangleleft O(1) \times \mathbb{R}$  shows that  $\text{Isom}(\mathbb{R})$  is solvable. Similarly in  $\mathbb{R}^2$ ,  $\text{Isom}(\mathbb{R}^2) = O(2) \times \mathbb{R}^2$  and the normal series  $\{1\} \triangleleft \mathbb{R}^2 \triangleleft SO(2) \times \mathbb{R}^2 \triangleleft O(2) \times \mathbb{R}^2$  shows that  $\text{Isom}(\mathbb{R}^2)$  is solvable. By Proposition 2.6,  $\text{Isom}(\mathbb{R})$  and  $\text{Isom}(\mathbb{R}^2)$  are amenable and hence  $\mathbb{R}$  and  $\mathbb{R}^2$  are not finitely paradoxical with respect to their isometry group.

For countable discrete group, there are two nice characterizations of amenability.

**Theorem 2.8.** *Suppose  $G$  is countable. Then the following are equivalent:*

- (i)  $G$  is amenable.
- (ii)  $G$  satisfies the Reiter's property: there exists a sequence of nonnegative functions  $f_n \in l^1(G)$  such that  $\|f_n\|_1 = 1$  for all  $n$ , and  $\lim_{n \rightarrow \infty} \|g \cdot f_n - f_n\|_1 = 0$  for all  $g \in G$ .
- (iii)  $G$  admits a Følner sequence, i.e. a sequence of finite subsets  $F_n \subset G$  such that  $\lim_{n \rightarrow \infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$  for all  $g \in G$ .

*Proof.* (i)  $\implies$  (ii): Let  $K = \{f \in l^1(G) : \|f\|_1 = 1, f \geq 0\}$ , and fix a mean  $m \in l^\infty(G)^*$ . We know  $m$  is in the unit ball of  $l^\infty(G)^*$  by the remark before Proposition 2.4. We will first show  $m$  is in the weak\* closure of  $K$ . Suppose not. As  $\overline{K}^{wk^*}$  is compact convex by Banach-Alaoglu theorem, by Hahn-Banach separation theorem there exists  $\phi_0 \in l^\infty(G)$  separating  $m$  from  $K$ , i.e. for all  $f \in K$ ,  $\text{Re}\langle \phi_0, f \rangle \leq \text{Re}\langle \phi_0, m \rangle - \epsilon$  for some  $\epsilon > 0$ . By replacing  $\phi_0$  with  $\text{Re} \phi_0$ , we may assume  $\phi_0$  is real-valued. Note that  $\|\phi_0\|_\infty \geq \phi_0(g_0) \geq \|\phi_0\|_\infty - \epsilon/2$  for some  $g_0 \in G$ . If  $f = \chi_{\{g_0\}}$ , then since  $\|m\| \leq 1$

$$\|\phi_0\|_\infty \geq m(\phi_0) = \langle \phi_0, m \rangle \geq \langle \phi_0, f \rangle + \epsilon = \phi_0(g_0) + \epsilon \geq \|\phi_0\|_\infty + \epsilon/2,$$

a contradiction. Hence  $m \in \overline{K}^{wk^*}$ . It follows that there is a net  $f_i \rightarrow m$  in the weak\* topology with  $f_i \in K$ , i.e. for all  $\phi \in l^\infty(G)$ ,  $\langle f_i, \phi \rangle \rightarrow m(\phi)$ . Next we observe that  $g \cdot f_i \rightarrow g \cdot m$  in the weak\* topology:

$$\begin{aligned} \sum_{h \in G} g \cdot f_i(h) \phi(h) &= \sum_{h \in G} f_i(g^{-1}h) \phi(h) = \sum_{h \in G} f_i(h) \phi(gh) \\ &= \sum_{h \in G} f_i(h) g^{-1} \cdot \phi(h) = \langle f_i, g^{-1} \cdot \phi \rangle \rightarrow m(g^{-1} \cdot \phi) = g \cdot m(\phi). \end{aligned}$$

By left-invariance of  $m$ ,  $g \cdot f_i \rightarrow m = g \cdot m$ . It then follows that  $f_i - g \cdot f_i \rightarrow 0$  in the weak\* topology. Since the counting measure is  $\sigma$ -finite,  $(l^1(G))^* \simeq l^\infty(G)$ . It follows that the weak\* topology on  $l^1(G)$  when viewed as a subspace of  $(l^\infty(G))^*$  is the same as the weak topology. Hence  $f_i - g \cdot f_i \rightarrow 0$  weakly.

Now fix  $\epsilon > 0$ . Enumerate  $G = \{g_n\}_{n \geq 1}$ . Fix  $n \geq 1$  and consider the convex subset

$$C = \{(f - g_1 \cdot f, f - g_2 \cdot f, \dots, f - g_n \cdot f) : f \in K\}$$

of the Banach space  $l^1(G)^{\oplus n}$  with the norm  $\|(f_1, f_2, \dots, f_n)\| = \sum_{i=1}^n \|f_i\|_1$ . Then  $0 \in \overline{C}^{wk}$ . By Mazur's theorem,  $\overline{B}^{wk} = \overline{B}^{\|\cdot\|}$  for any convex subset  $B$  of a Banach space. So,  $0 \in \overline{C}^{\|\cdot\|}$  and therefore there exists  $f_n \in l^1(G)$  with  $\|f_n\|_1 = 1$  and  $f_n \geq 0$  such that  $\sum_{i=1}^n \|f_n - g_i \cdot f_n\| < \frac{1}{n}$ .

(ii)  $\implies$  (iii): If  $\psi, \phi \in l^1(G)$ , and  $\psi, \phi \geq 0$ , then by Tonelli's theorem

$$\|\psi\|_1 = \int_0^\infty \|\chi_{\{\psi > t\}}\|_1 dt \quad \text{and} \quad \|\psi - \phi\|_1 = \int_0^\infty \|\chi_{\{\psi > t\}} - \chi_{\{\phi > t\}}\|_1 dt.$$

By the Reiter property, for any  $n \geq 1$ , there exists  $f \in l^1(G)$  with  $f \geq 0$ ,  $\|f\|_1 = 1$  such that  $\sum_{i=1}^n \|f - g_i \cdot f\|_1 < \frac{1}{n}$ . Let  $X_t = \{h \in G : f(h) > t\}$  for  $t > 0$ .  $X_t$  is finite since  $f \in l^1(G)$ . Also note that  $\{h \in G : g \cdot f(h) > t\} = g^{-1}X_t$  so  $\|\chi_{\{f>t\}} - \chi_{\{g \cdot f>t\}}\|_1 = |g^{-1}X_t \Delta X_t|$  for all  $g \in G$ . Combining with the above equation, we get

$$\begin{aligned} \sum_{i=1}^n \|f - g_i \cdot f\|_1 &= \sum_{i=1}^n \int_0^\infty \|\chi_{\{f>t\}} - \chi_{\{g_i \cdot f>t\}}\|_1 dt \\ &= \sum_{i=1}^n \int_0^\infty |g_i^{-1}X_t \Delta X_t| dt < \frac{1}{n} = \frac{1}{n} \|f\|_1 = \int_0^\infty \frac{|X_t|}{n} dt. \end{aligned}$$

Hence there exists  $\{k_n\}$  such that  $F_n = X_{k_n}$  satisfies  $\sum_{i=1}^n |g_i^{-1}F_n \Delta F_n| < \frac{|F_n|}{n}$ .

(iii)  $\implies$  (i): Let  $\{F_n\}_{n \geq 1}$  be a Følner sequence, and let  $\omega \in \beta\mathbb{N}$  be a free ultrafilter, and observe that

$$\mu(A) = \lim_{n \rightarrow \omega} \frac{|A \cap F_n|}{|F_n|}$$

defines a finitely additive probability measure on  $G$ . Note that

$$\begin{aligned} |\mu(gA) - \mu(A)| &= \lim_{n \rightarrow \omega} \frac{||gA \cap F_n| - |A \cap F_n||}{|F_n|} \\ &= \lim_{n \rightarrow \omega} \frac{||A \cap g^{-1}F_n| - |A \cap F_n||}{|F_n|} \\ &\leq \lim_{n \rightarrow \omega} \frac{|g^{-1}F_n \Delta F_n|}{|F_n|} \\ &= 0. \end{aligned}$$

Therefore,  $\mu$  is left-invariant. □

*Remark 2.9.* For uncountable discrete groups, one can generalize with Følner nets replacing Følner sequences. In general, a discrete group  $G$  is amenable iff  $G$  satisfies the Følner condition: for any finite subset  $\Gamma \subset G$  and  $\epsilon > 0$ , there exists finite subset  $F \subset G$  such that for any  $g \in \Gamma$ ,  $\frac{|gF \Delta F|}{|F|} < \epsilon$ .

**Example 2.10.**  $F_n = [-n, n]$  for  $n \in \mathbb{N}$  is a Følner sequence for  $\mathbb{Z}$ .

In Example 2.7, we showed that there is a measure on  $\text{Isom}(\mathbb{R})$  and  $\text{Isom}(\mathbb{R}^2)$  so by Proposition 2.3 there is a finitely additive isometry-invariant probability measure on  $\mathbb{R}$  and  $\mathbb{R}^2$ . With the following equivalent formulation of amenability in terms of the Hahn-Banach extension property, we can show in fact there exists a finitely additive isometry-invariant measure that extends a particularly nice one, the Lebesgue measure.

**Definition 2.11** (Hahn-Banach extension property). A group  $G$  satisfies the Hahn-Banach extension property if  $G$  is a group of linear operators on a real vector space  $V$ ,  $F$  is a  $G$ -invariant linear functional on a  $G$ -invariant subspace  $V_0$  of  $V$ , and  $F$  is dominated by a sublinear functional such that  $p(g(v)) \leq p(v)$  for  $g \in G$  and  $v \in V$ , then there exists a  $G$ -invariant linear functional  $\tilde{F}$  on  $V$  that extends  $F$  and is dominated by  $p$ .

**Theorem 2.12.**  $G$  is amenable iff  $G$  satisfies the Hahn-Banach extension property.

*Proof.* Suppose  $G$  is amenable with measure  $\mu$ . By Hahn-Banach theorem, there exists a linear functional  $F_0$  on  $V$  that extends  $F$  and is dominated by  $p$ . Then for any  $v \in V$ , define  $f_v : G \rightarrow \mathbb{R}$  by  $f_v(h) = F_0(h^{-1}v)$ . Since  $F_0(h^{-1}v) \leq p(h^{-1}v) \leq p(v)$ ,  $f_v$  is dominated by  $p(v)$ . Define  $\tilde{F}(v) = \int f_v d\mu$ . Then  $\tilde{F}(v) \leq p(v)$  and  $\tilde{F}$  is a linear functional on  $V$ . Moreover,  $\tilde{F}$  extends  $F$ , and the  $G$ -invariance follows from that of  $\mu$ . Conversely, let  $V = l^\infty(G)$ , the collection of bounded real-valued functions on  $G$ , and let  $V_0$  be the subspace of constant functions. The action of  $G$  on  $l^\infty(G)$  given by  $g.f(h) = f(g^{-1}h)$  is linear, and  $V_0$  is  $G$ -invariant. Let  $F(\alpha\chi_G) = \alpha$  and  $p(f) = \sup\{f(g) : g \in G\}$ . Then by the Hahn-Banach extension property, there is a left-invariant linear function  $\tilde{F}$  on  $l^\infty(G)$  dominated by  $p$  with  $\tilde{F}(\chi_G) = 1$ . If  $f \geq 0$ , then  $p(-f) \leq 0$  and so  $\tilde{F}(-f) \leq p(-f) \leq 0$ . Hence  $\tilde{F}(f) = -\tilde{F}(-f) \geq 0$ . This shows that  $\tilde{F}$  is positive. Therefore,  $\tilde{F}$  is a left-invariant mean on  $G$  and  $G$  is amenable by Proposition 2.4.  $\square$

**Corollary 2.13.** If  $G$  is an amenable group of isometries of  $\mathbb{R}^n$ , then there is a finitely additive,  $G$ -invariant extension of the Lebesgue measure to all subsets of  $\mathbb{R}^n$ . In particular, the Lebesgue measure on  $\mathbb{R}$ , or  $\mathbb{R}^2$  has an isometry-invariant, finitely additive extension to all sets.

*Proof.* Suppose  $G$  is an amenable group of  $\text{Isom}(\mathbb{R}^n)$ . Let  $V_0$  be the space of all Lebesgue integrable real-valued functions on  $\mathbb{R}^n$ , and let  $V$  be the space of all functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for some  $g \in V_0$ ,  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}^n$ .  $\text{Isom}(\mathbb{R}^n)$  acts on  $V$  and  $V_0$  in the obvious way and we let  $F$  be the  $G$ -invariant linear functional on  $V_0$  defined by  $F(f) = \int f dm$ . Finally, define a  $G$ -invariant sublinear functional  $p$  on  $V$  that dominates  $F$  by  $p(f) = \inf\{F(g) : g \in V_0, g \geq f\}$ . By the Hahn-Banach extension property, there exists a  $G$ -invariant linear functional  $\tilde{F}$  on  $V$ , and define the desired measure  $\mu$  on  $\mathcal{P}(\mathbb{R}^n)$  by  $\mu(A) = \tilde{F}(\chi_A)$  if  $\chi_A \in V$ ;  $\mu(A) = \infty$  otherwise. It is easy to see that  $\mu$  is finitely additive and  $G$ -invariant. Since  $\tilde{F}(\chi_A) = \int \chi_A dm = m(A)$  if  $A$  has finite Lebesgue measure,  $\mu$  extends  $m$ . Finally, note that  $\mu(A) \geq 0$ ; for if  $f \geq 0$ , then  $\tilde{F}(-f) \leq p(-f) \leq 0$  and  $\tilde{F}(f) = -\tilde{F}(-f) \geq 0$ .  $\square$

*Remark 2.14.* Recall from Corollary 1.2 that there is no countably additive translation-invariant measure on all subsets of  $\mathbb{R}^n$  that normalizes the cube. But the group of translations is abelian and hence amenable by Proposition 2.6, so if only finite rather than countable additivity is desired, then such measure do exists by the preceding corollary.

**Corollary 2.15.** If  $G$  is an amenable group of isometries of  $\mathbb{R}^n$ , then no bounded subset of  $\mathbb{R}^n$  with nonempty interior is  $G$ -paradoxical. In particular, no bounded subset of  $\mathbb{R}$  or  $\mathbb{R}^2$  with nonempty interior is finitely paradoxical.

*Proof.* By the preceding corollary, there exists a finitely additive  $G$ -invariant extension of  $m$  to all subsets of  $\mathbb{R}^n$ . Then  $0 < \mu(A) < \infty$  for any bounded set  $A$  with nonempty interior, so  $A$  is not  $G$ -paradoxical.  $\square$

We've seen in Proposition 2.6 that most groups are amenable. We also know from Corollary 1.10 that any group with a free subgroup of rank 2 is not amenable. Day conjectured that this is the only way a group can fail to be amenable and this became known as the von Neumann conjecture. This was unsolved until Ol'shanskii in 1980 showed that the conjecture is false by constructing an example of a torsion

paradoxical group. A torsion group has no element of infinite order and therefore has no free subgroup of any rank. Nevertheless, Tits showed that the conjecture is true for linear groups, i.e. subgroups of  $GL_n(\mathbb{F})$  for some field  $\mathbb{F}$ . See Wagon [2] for details on Tits alternative.

If a group has a subgroup of finite index with a certain property, we shall say that the group virtually has the property.

**Theorem 2.16** (Tits alternative).  $G \subset GL_n(\mathbb{F})$ .

- (i) If  $\mathbb{F}$  has characteristic 0, then either  $G$  has a free subgroup of rank 2 or  $G$  is virtually solvable.
- (ii) If  $\mathbb{F}$  has nonzero characteristic, then either  $G$  has a free subgroup of rank 2 or  $G$  has a normal solvable subgroup  $H$  such that  $G/H$  is locally finite, i.e. every finite subset generates a finite subgroup.

In particular, Tits' theorem applies to any group of Euclidean isometries since the Euclidean affine group  $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$  is isomorphic to a subgroup of  $GL_{n+1}(\mathbb{R})$ .

**Corollary 2.17.** Let  $G \subset \text{Isom}(\mathbb{R}^n)$ . Then  $G$  is amenable iff  $G$  has no free subgroup of rank 2.

### 3. SUPRAMENABILITY AND GROWTH CONDITIONS IN GROUPS

In this final section we shall present some connections between the amenability of a group and the growth rate of a group, i.e., the speed at which new elements appear when one considers longer and longer words using letters from a fixed finite subset of a group. This approach to amenability sheds light on a basic difference between abelian and solvable groups. Both types of groups are amenable, but their growth rates can be quite different. This will explain why there is a paradoxical subset of the plane by Sierpiński-Mazurkiewicz paradox, but no similar subset of  $\mathbb{R}$  exists. The notion of amenability is based on the existence of a probability measure, but we are often interested in invariant measure that assign specific subsets measure one. The following definition is the appropriate strengthening of amenability that guarantees the existence of such measures for any nonempty subset of a set on which the group acts.

**Definition 3.1** (Supramenability). A group  $G$  is supramenable if for any nonempty subset  $A \subset G$ , there exists a finitely additive, left-invariant measure  $\mu : \mathcal{P}(G) \rightarrow [0, \infty]$  with  $\mu(A) = 1$ .

Of course, if  $G$  is supramenable, then no nonempty subset of  $G$  is paradoxical. The converse is true by Tarski's theorem.

**Theorem 3.2** (Tarski's theorem).  $G$  is supramenable iff no nonempty subset of  $G$  is finitely paradoxical.

The main result on actions of such groups is that supramenability of the group can be transferred to the set on which the group acts.

**Proposition 3.3.** Let  $G$  be a supramenable group acting on  $X$ , and  $A$  is a nonempty subset of  $X$ . Then there is a finitely additive,  $G$ -invariant measure  $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$  such that  $\mu(A) = 1$ ; hence no nonempty subset of  $X$  is finitely  $G$ -paradoxical.

*Proof.* Fix  $x \in A$ . Define  $G_B = \{g \in G : g.x \in B\}$  for every  $B \subset X$ . Note that  $1 \in G_A$ . By supramenability of  $G$ , there exists a finitely additive, left-invariant measure  $\nu : \mathcal{P}(G) \rightarrow [0, \infty]$  with  $\nu(G_A) = 1$ . Define the desired measure  $\mu$  on  $\mathcal{P}(X)$  by  $\mu(B) = \nu(G_B)$ . Then  $\mu(A) = \nu(G_A) = 1$  and the finite additivity of  $\nu$  follows from that of  $\mu$ . Finally, if  $g \in G$ ,  $G_{g.B} = g.G_B$ , whence the  $G$ -invariance of  $\mu$  follows from that of  $\nu$ .  $\square$

Of course, any nonamenable group fails to be supramenable, but the Sierpiński-Mazurkiewicz paradox provides an example of an amenable group that is not supramenable. In Example 2.7, we showed that  $\text{Isom}(\mathbb{R}^2)$  is amenable, but the existence of a paradoxical set of the plane and the preceding proposition implies  $O(2) \times \mathbb{R}^2$  is not supramenable. This can be shown more directly by constructing a free subsemigroup of rank 2 in  $O(2) \times \mathbb{R}^2$  as in Theorem 1.13. In fact, free semigroups of rank 2 play much the role for supramenability that free groups of rank 2 do for amenability. As we now show, a group that contains a free subsemigroup of rank 2 can not be supramenable.

**Proposition 3.4.** If  $G$  contains a free subsemigroup of rank 2, then  $G$  is not supramenable.

*Proof.* Let  $S \subset G$  be the semigroup generated by  $g$  and  $h$ . Observe that  $gS$  and  $hS$  are disjoint subsets of  $S$  that are clearly finitely  $G$ -equidecomposable with  $S$ . Hence,  $S$  is finitely  $G$ -paradoxical.  $\square$

Now we summarize the known closure properties of supramenable groups.

**Proposition 3.5.**

- (i) Every finite group is supramenable.
- (ii) If  $G$  is supramenable and  $\Gamma \leq G$ , then  $\Gamma$  is supramenable.
- (iii) If  $N \triangleleft G$  and  $G$  is supramenable then  $G/N$  is supramenable.
- (iv) (AC) If  $G$  is the direct union of a directed system of supramenable groups  $\{G_\alpha : \alpha \in I\}$ , then  $G$  is supramenable.
- (v) If  $H \leq G$  and  $[G : H] < \infty$ , then  $H$  is supramenable implies  $G$  is supramenable.

*Proof.*

- (i) If  $A \subset G$  is nonempty, let  $\mu$  be defined by  $\mu(B) = |B|/|A|$ .
- (ii) Let  $\Gamma$  be a subgroup of a supramenable group  $G$  and let  $A$  be a nonempty subset of  $\Gamma$ . Simply restrict a measure on  $\mathcal{P}(G)$  that normalizes  $A$  to  $\mathcal{P}(\Gamma)$ .
- (iii) If  $A$  is a nonempty subset of  $G/N$ , let  $\mu$  be a left-invariant measure on  $\mathcal{P}(G)$  that normalizes  $\cup A$ . Define  $\nu$  on  $\mathcal{P}(G/N)$  by  $\nu(B) = \mu(\cup B)$ .
- (iv) Let  $A$  be a nonempty subset of  $G$ . Since each  $G_\alpha$  is contained in a  $G_\beta$  that intersects  $A$ , we may assume without loss of generality that each  $G_\alpha$  intersect  $A$  by simply deleting from the system any subgroups that miss  $A$ . Consider the topological space  $[0, \infty]^{\mathcal{P}(G)}$ , which is compact by Tychonoff's theorem. For each  $\alpha \in I$ , let  $\mathcal{M}_\alpha$  consist of those finitely additive  $\mu : \mathcal{P}(G) \rightarrow [0, \infty]$  such that  $\mu(A) = 1$  and  $\mu$  is  $G_\alpha$ -invariant. Each  $G_\alpha$  is supramenable, so if  $\mu_\alpha$  is a  $G_\alpha$ -invariant measure on  $\mathcal{P}(G_\alpha)$  with  $\mu_\alpha(A \cap G_\alpha) = 1$ , then the measure defined by  $\mu(B) = \mu_\alpha(B \cap G_\alpha)$  lies in  $\mathcal{M}_\alpha$ . So  $\mathcal{M}_\alpha \neq \emptyset$ . As in Proposition 2.6(v),  $\mathcal{M}_\alpha$  is closed and  $\mathcal{M}_\alpha \cap \mathcal{M}_\beta \supset \mathcal{M}_\gamma$  if  $G_\alpha, G_\beta \subset G_\gamma$  implies  $\{\mathcal{M}_\alpha : \alpha \in I\}$  has the finite intersection property. By compactness, there exists  $\mu \in \cap_{\alpha \in I} \mathcal{M}_\alpha$  and such a  $\mu$  is  $G$ -invariant measure normalizing  $A$ .



- (v) Let  $A$  be a nonempty subset of  $G$  and  $\{g_1, \dots, g_m\}$  be right coset representatives of  $H$  in  $G$ . Since  $H$  is supramenable, apply Theorem 3.3 to the action of  $H$  on  $G$  by left multiplication to obtain an  $H$ -invariant measure  $\nu$  on  $\mathcal{P}(G)$  with  $\nu(\cup g_i A) = 1$ . Note that  $0 < \sum \nu(g_i A) < \infty$ . Define  $\mu$  on  $\mathcal{P}(G)$  by  $\mu(B) = \frac{1}{\sum \nu(g_i A)} \sum \nu(g_i B)$ . Then  $\mu$  is finitely additive and  $\mu(A) = 1$ . Moreover, for any  $g \in G$ , the set  $\{g_i g\}$  is a set of representative of the right cosets of  $H$ . Hence  $\mu(gB) = \frac{1}{\sum \nu(g_i A)} \sum \nu(g_i g B) = \frac{1}{\sum \nu(g_i A)} \sum_k \nu(h_k g_k B)$ , and the  $H$ -invariance of  $\nu$  then yields that  $\mu(gB) = \mu(B)$ .  $\square$

We now discuss growth rates in groups and how they relate to supramenability. Recall that the length of a reduced word  $g_1^{m_1} \cdots g_r^{m_r}$  ( $g_i$  not necessarily distinct,  $g_i \neq g_{i+1}$ ,  $m_i > 0$ ) is  $\sum m_i$ ; the identity is assumed to have length 0.

**Definition 3.6** (Growth rate of a group). If  $S$  is a finite subset of a group  $G$ , then the growth function  $\gamma_S : \mathbb{N} \rightarrow \mathbb{N}$  is defined by setting  $\gamma_S(n)$  equal to the number of elements of  $G$  obtainable as a reduced word of length at most  $n$  using elements of  $S \cup S^{-1}$  as letters.

$\gamma_S$  is of course nondecreasing. Also,  $\gamma_S(0) = 1$  and  $\gamma_S(1) = |\{1\} \cup S \cup S^{-1}|$ . Since  $\gamma_S(n+m) \leq \gamma_S(n)\gamma_S(m)$ , it follows that  $\gamma_S(n) \leq \gamma_S(1)^n$ , so  $\gamma_S$  is always bounded by an exponential function. If  $G$  contains a free subsemigroup of rank 2, and  $S$  contains two free generators of such a semigroup, then  $\gamma_S(n) \geq 2^n$ , the number of words in  $S$  with only positive exponents and with length exactly  $n$ . Hence if  $G$  contains a free subsemigroup of rank 2, then the growth function exhibits exponential growth, with respect to some choice of  $S$ .

**Definition 3.7.** A group  $G$  has subexponential growth if for any finite  $S \subset G$  and any  $b > 1$ , there exists  $N > 0$  such that  $\gamma_S(n) < b^n$  for  $n \geq N$ ; equivalently,  $\lim_{n \rightarrow \infty} \gamma_S(n)^{1/n} = 1$ . Otherwise  $G$  is said to have exponential growth.

Before we give examples of groups of subexponential growth, we should first point the important connection with supramenability.

**Theorem 3.8.**

- (i) Let  $G$  acts on  $X$  and let  $A$  be a nonempty subset of  $X$ . If  $G$  has subexponential growth, then  $A$  is not finitely  $G$ -paradoxical.  
(ii) (AC) If  $G$  has subexponential growth, then  $G$  is supramenable.

*Proof.*

- (i) Suppose on the contrary that  $A$  is finitely  $G$ -paradoxical. Then there exists two injective piecewise  $G$ -transformation  $F_1 : A \rightarrow A$  and  $F_2 : A \rightarrow A$  such that  $F_1(A) \cap F_2(A) = \emptyset$ . Let  $S$  be the set of all elements of  $G$  occurring as a part of the  $G$ -transformation in  $F_1$  and  $F_2$ . Since  $G$  has subexponential growth, there is an integer  $n$  such that  $\gamma_S(n) < 2^n$ . Consider the  $2^n$  functions  $\{H_i : 1 \leq i \leq 2^n\}$  obtained as compositions of a string of  $n$  functions that are either  $F_1$  or  $F_2$ , so each  $H_i$  looks like  $F_1 \circ F_2 \circ F_2 \circ \cdots \circ F_2 \circ F_1$ . Suppose  $i \neq j$ . Let  $k$  be the first (rightmost) of the  $n$  positions where  $H_i$  and  $H_j$  differ. Since the function obtained by restricting  $H_i$  and  $H_j$  to the rightmost  $k-1$  compositions is injective, and since  $F_1$  and  $F_2$  map  $A$  into disjoint sets,  $H_i(A) \cap H_j(A) = \emptyset$ . Then for any  $x \in A$ , the set  $\{H_i(x) : 1 \leq i \leq 2^n\}$  has

$2^n$  elements. But each  $H_i(x)$  has the form  $wx$  where  $w$  is a word of length  $n$  composed of elements of  $S$ . This contradicts the fact that  $\gamma_S(n) < 2^n$ .

- (ii) Consider the left translation action of  $G$  on itself. By part (i),  $G$  contains no nonempty finitely paradoxical subset. By Tarski's theorem,  $G$  is supra-amenable.

□

**Proposition 3.9.**

- (i) Every finite group has subexponential growth.
- (ii) Every abelian group has subexponential growth.
- (iii) A subgroup or a homomorphic image of a group of subexponential growth has subexponential growth.
- (iv) If  $H \leq G$  and  $[G : H] < \infty$ , then  $H$  has subexponential growth implies  $G$  has subexponential growth.
- (v) A direct union of groups of subexponential growth has subexponential growth; in particular, a group has subexponential growth iff all of its finitely generated subgroups are.

*Proof.* (i) For any  $S \subset G$ ,  $\gamma_S(n) \leq |G|$  so  $\gamma_S(n)^{1/n} \rightarrow 1$ .

- (ii) Suppose  $G$  is abelian. Let  $S = \{g_1, \dots, g_r\} \subset G$ . Any word in  $S$  is of the form  $g_1^{m_1} g_2^{m_2} \dots g_r^{m_r}$ , where  $m_i \in \mathbb{Z}$ . Hence the number of group elements that arise from words of length  $n$  is at most  $(2n+1)^r$  since  $-n \leq m_i \leq n$  for every  $1 \leq i \leq r$ . Thus  $\gamma_S(n)$  is dominated by  $n(2n+1)^r$ , a polynomial of degree  $r+1$ . This shows that abelian groups have subexponential growth.
- (iii) The subgroup case follows from the fact that  $\gamma_S$  with respect to  $H \leq G$  is bounded by  $\gamma_S$  with respect to  $G$ . Let  $N \triangleleft G$  and let  $S$  be a finite subset of  $G/N$ . Let  $S'$  be a finite subset of  $G$  whose image under  $G \rightarrow G/N$  is  $S$ . Then  $\gamma_S(n)$  in  $G/N$  is bounded by  $\gamma_{S'}(n)$  in  $G$ .
- (iv) Let  $g_1, \dots, g_r$  be a set of representative for the right cosets of  $H$  in  $G$  with  $g_1 = 1$ . Given a finite  $S \subset G$ , let  $S'$  consists of the finitely many  $h \in H$  that arises when each  $g_i s$  is written in the form  $h g_k$ , where  $s \in S$  and  $1 \leq i \leq r$ . Suppose  $w = s_1 s_2 \dots$  with  $s_i \in S$  is a word of length at most  $n$  from  $S$ . Since  $s_1 = 1 s_1 = g_1 s_1$ , there exists  $h_1 \in H$  and  $1 \leq k_1 \leq r$  such that  $s_1 = h_1 g_{k_1}$ . Similarly, there exists  $h_2 \in H$  and  $1 \leq k_2 \leq r$  such that  $g_{k_1} s_2 = h_2 g_{k_2}$ . By induction,  $w = h_1 \dots h_m g_{k_m}$ , where  $m \leq n$  is the length of  $w$ . So, each  $w \in G$  that arises as a word of length at most  $n$  from  $S$  can be represented as  $w' g_i$ , where  $1 \leq i \leq r$  and  $w'$  is a word of length at most  $n$  from  $S'$ . Therefore,  $\gamma_S(n)$  in  $G$  is at most  $r \cdot \gamma_{S'}(n)$  in  $H$ . It follows that  $\lim \gamma_S(n)^{1/n} \leq \lim r^{1/n} \gamma_{S'}(n)^{1/n} = 1$ , so  $G$  has subexponential growth.
- (v) Suppose  $G$  is the union of the directed system of subgroups  $\{G_\alpha : \alpha \in I\}$  and  $G$  has exponential growth. If this failure is witnessed by  $\gamma_S$  then choose  $\alpha \in I$  such that  $S \subset G_\alpha$ . Since  $\gamma_S$  only refers to elements in the group generated by  $S$ ,  $G_\alpha$  must have exponential growth too.

□

In the above proposition, we have explained why abelian groups has subexponential growth. Combined with Theorem 3.8, we know that abelian groups are supra-amenable. In particular, this yields a proof that abelian groups are amenable that is completely different from the proof of Proposition 2.6. This new proof using growth rate is somewhat more informative, because in addition to proving the

stronger conclusion regarding supramenability, it shows quite clearly and effectively why the existence of a finitely  $G$ -paradoxical set in any action of  $G$  implies that  $G$  has exponential growth.

**Corollary 3.10.** The isometry group of the line  $\text{Isom}(\mathbb{R}) = O(1) \times \mathbb{R}$  has subexponential growth. Therefore no nonempty subset of  $\mathbb{R}$  is finitely paradoxical.

*Proof.* Note that  $\mathbb{R} \triangleleft \text{Isom}(\mathbb{R})$  and  $\text{Isom}(\mathbb{R})/\mathbb{R} \simeq O(1) \simeq \mathbb{Z}/2\mathbb{Z}$ . Since  $\mathbb{R}$  is abelian,  $\text{Isom}(\mathbb{R})$  has subexponential growth by Proposition 3.9 (ii) and (iv). The conclusion then follows from Theorem 3.8 (i).  $\square$

So, in summary no nonempty subset of  $\mathbb{R}$  is finitely paradoxical because  $\text{Isom}(\mathbb{R})$  is supramenable. But there exists paradoxical subset of  $\mathbb{R}^2$  because  $SO(2) \times \mathbb{R}^2$  contains a free subsemigroup of rank 2 and hence not supramenable; an example is the Sierpiński-Mazurkiewicz paradox. As a consequence of the amenability of  $\text{Isom}(\mathbb{R})$  and  $\text{Isom}(\mathbb{R}^2)$ , Lebesgue measure on  $\mathbb{R}$  and  $\mathbb{R}^2$  has a finitely additive, isometry-invariant extension to all sets. However,  $\text{Isom}(\mathbb{R}^2)$  is solvable and hence amenable so all of  $\mathbb{R}^2$  is not finitely paradoxical. In  $\mathbb{R}^3$  however,  $SO(3)$  is not amenable because it contains a free subgroup of rank 2. This lead to the Banach-Tarski paradox that  $S^2$  is finitely  $SO(3)$ -paradoxical and  $\mathbb{R}^3$  is finitely paradoxical. In fact, as shown in Theorem 1.20, all bounded sets of  $\mathbb{R}^3$  with nonempty interior are finitely equidecomposable. Banach-Tarski paradox also exists in  $\mathbb{R}^n$  for all  $n \geq 3$ :  $S^{n-1}$  is finitely  $SO(n)$ -paradoxical and  $\mathbb{R}^n$  is finitely paradoxical when  $n \geq 3$ .

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