Chern Classes and Lines on a Cubic Surface

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Abstract

In this thesis, we develop the basic theory of Chern classes in the setting of algebraic geometry. We motivate the definitions using the more classical picture of vector bundles in topology. We discuss some of the analogies between the theory in the topological setting and the algebro-geometric setting. We also make explicit several identifications that are often made in this area. As a demonstration of the theory, we show that the top Chern class of a vector bundle encodes information about the zero locus of a generic section and we use that interpretation to compute the number of lines on smooth cubic surfaces in $\mathbb{P}^3$.

Contents

Introduction

The idea of assigning a vector to each point in a space is a very intuitive one, and this is often done when one models fluid flow, electric fields, or other physical phenomena. Mathematically, since vectors must live in a vector space, implicit in this assignment is the fact that we are also attaching whole vector spaces to each point of our space. The ways in which one may attach vector spaces to a fixed base space can be studied, and knowledge of possible ways of attaching tells us something about the geometry of the base space. This is the perspective taken in *K-theory*. A seemingly simple question is whether a given manner of attaching vector spaces to a base space is "twisted" or not. As it turns out, answering this question in general is very difficult. In some sense, *Chern classes* measure how "twisted" the manner of attaching is.

It turns out that this geometric discussion can be realized in the setting of algebraic geometry. In this context, Chern classes can be used to solve problems in enumerative geometry. In this thesis, we will develop the theory of Chern classes to the extent that we can reasonably answer the following question:

How many lines are on a smooth cubic surface in $\mathbb{P}^3$*?*

This is a question that is asking about the nature of the intersection between lines (which are the zero loci of degree 1 polynomials) and cubics surfaces (which are the zero loci of degree 3 polynomials). The theory of Chern classes is robust enough to solve several other similar enumerative problems. To develop Chern classes, one must develop several notions in intersection theory. In particular, we will spend a lot of time discussing idea of *multiplicity*. This will make it possible for us to create an algebro-geometric analog of homology. Chern classes in algebraic geometry are operators on this analog of homology.

We will begin by defining *vector bundles* in Chapter 1, which are the mathematically precise way of attaching vector spaces to some base space while respecting the geometry of the base space. We will make precise what mean for a vector bundle to be "twisted" and we will illustrate that it is difficult to determine whether this is the case or not in general. We will then adapt many of our geometric constructions to algebraic geometry. In Chapter 2, we will partially set up *Chow rings*, which are the analog of homology in algebraic geometry. In Chapter 3, we define the Chern classes in the setting of algebraic geometry and briefly discuss the counterparts in the setting of topology. We conclude with Chapter 4 where we apply our theory to answer our enumerative question stated above.

This thesis is purely expository and primarily based on [7], [10], and MATH 203 taught over the 2022–2023 academic year at UC San Diego. Setting up all of these objects becomes rather technical and there is a limit to how much we can state here. We will often defer to other sources, but we will attempt to at least state when our methods are insufficient. In some cases, we will establish some results which are best proven using methods which we will not develop here, such as in Proposition 5 where we will briefly allude to deformation theory. Several sources often leave out details and the author has attempted to fill them in or provide alternate references whenever possible.

We will assume the reader has some knowledge of algebraic geometry (especially scheme theory), algebraic topology, analysis, and commutative algebra. In some sense, the list of prerequisite material is completely arbitrary and is chosen based off of the desire for a relatively self-contained paper, time constraints, and the background knowledge of the author at the time of writing.

Chapter 1

Vector Bundles

We begin with the definition of a *vector bundle*, which is the basic geometric object which we will study and use.

Definition 1. *Let X be a topological space and k be a field endowed with a topology. A* ***k-vector bundle of rank*** *r is a topological space E along with a continuous surjection $\pi \colon E \to X$ such that*

(i) For every $x \in X$ the fiber $\pi^{-1}(\{p\})$ is endowed with the structure of a k-vector space of dimension r.

(ii) For every $x \in X$ there is a neighborhood $U \subseteq X$ of x and a homeomorphism $\Phi_U \colon \pi^{-1}(U) \to U \times k^r$ such that $\pi|_{\pi^{-1}(U)} = \pi_U \circ \Phi_U$ where $\pi_U \colon U \times k^r \to U$ is the projection, and for every $y \in U$, the restriction of Φ_U to the fiber $\pi^{-1}(\{y\})$ is a vector space isomorphism. This map Φ_U is called a ***local trivialization****.*

In this paper, a vector bundle without a specified field will always be taken to mean a complex vector bundle, though much of the discussion will generalize to arbitrary fields. We will refer to the vector bundle $(E, \pi \colon E \to X)$ as simply either E or $\pi \colon E \to X$ as context permits.

Vector bundles are the rigorous way to encode the idea of vector spaces being attached to each point of a space. The most complicated axiom is the second one, which requires us to not just assign vector spaces randomly to each point in a space, but to do so in a way that makes the resulting object locally appear like a Cartesian product. We will now turn to the question of determining when a vector bundle is not just locally, but also globally homeomorphic to a Cartesian product.

1.1 Triviality

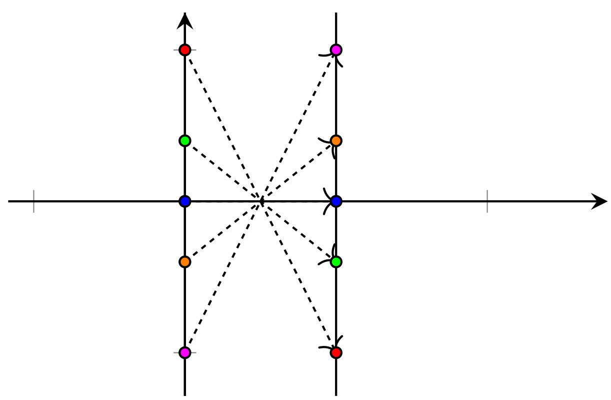
Of course, for any topological space X, the simplest possible complex vector bundle of rank r on X is the Cartesian product $X \times \mathbb{C}^r$ where $\pi \colon X \times \mathbb{C}^r \to X$ is just given by the projection map. In this case, we just naively attach a copy of $\mathbb{C}^r$ on top of each point of X in a way that is obviously compatible with the topology on X. In particular, for any $x \in X$, there is no need to be picky in choosing the neighborhood U of x for a local trivialization. One may just choose all of X to be the neighborhood of any point in X on which there is a local trivialization. If such a trivialization exists, one calls this a *global trivialization*, and the vector bundle is said to be *trivial*.

One may believe that every vector bundle is trivial. If in fact we are simply attaching vector spaces isomorphic to $\mathbb{C}^r$ (or $\mathbb{R}^n$ in the case of real vector bundles) to each point of X, it is not obvious that the bundle is not just $X \times \mathbb{C}^r$. In fact, vector bundles can be nontrivial, with nontriviality arising from the global topology of the space X.

For instance, one of the prototypical examples of a vector bundle is the *tangent bundle* on a smooth manifold. This is the vector bundle that attaches at each point of a smooth manifold the tangent space to the manifold at that point. It turns out that the sphere S^2 has a nontrivial tangent bundle. This is implied by the *hairy ball theorem* which states that there are no nonvanishing continuous vector fields on S^2.

If one wishes to avoid the machinery required to set up the smooth category for tangent bundles, one can also find an example of a nontrivial vector bundle on S^1. In particular, the *Möbius bundle* is a real vector bundle of rank 1 on S^1 that is nontrivial. We describe its construction as follows.

Define an equivalence relation $\sim$ on $\mathbb{R}^2$ by $(x, y) \sim (x', y')$ if and only if $(x', y') = (x+n, (-1)^n y)$ for some integer n. This equivalence relation identifies the points in the interior of the strip $[0, 1] \times \mathbb{R}$ with themselves while identifying points on one edge of the strip with a mirrored point on the other edge of the strip.



Let $E = \mathbb{R}^2/\sim$ and let $q\colon \mathbb{R}^2 \to E$ be the quotient map. Let $\varepsilon\colon \mathbb{R} \to S^1$ be the exponential map $\varepsilon(x) = e^{2\pi i x}$ (where we identify S^1 with the unit circle in the complex plane). Observe that for any $n \in \mathbb{Z}$ we have

$$(\varepsilon \circ \pi_1)(x, y) = \varepsilon(x) = e^{2\pi i x} = e^{2\pi i x} e^{2\pi i n} = e^{2\pi i (x+n)} = \varepsilon(x + n) = (\varepsilon \circ \pi_1)(x + n, (-1)^n y), \quad (1.1)$$

where π_1 is the projection onto the first factor. The above calculation establishes that if $(x, y) \sim (x', y')$, then $(\varepsilon \circ \pi_1)(x, y) = (\varepsilon \circ \pi_1)(x', y')$. Therefore, we may descend to a map $\varphi\colon E \to S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{q} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{R} & \xrightarrow[\varepsilon]{} & S^1 \end{array} \tag{1.2}$$

We claim that φ is a nontrivial vector bundle on S^1. This bundle is known as the *Möbius bundle*. First we check that this is a vector bundle on S^1. The surjectivity of the map is clear by the above diagram: every point in S^1 is of the form $e^{2\pi ix}$. For any $p \in S^1$, we have $\pi^{-1}(\{p\}) = \bigcup_{y\in\mathbb{R}} \overline{(x_p, y)}$, where $x_p \in [0, 1)$ is such that $\varepsilon(x_p) = p$. So $\pi^{-1}(\{p\})$ has a natural real vector space structure isomorphic to $\mathbb{R}$, where the isomorphism is given by $\overline{(x_p, y)} \mapsto y$.

ε is a covering map, so for $p \in S^1$ we can find a neighborhood U of p such that U is evenly covered by ε. Let $\widetilde{U}$ be a component of $\varepsilon^{-1}(U)$. Then q restricts to a homeomorphism onto its image on $\widetilde{U} \times \mathbb{R}$ by the definition of the quotient topology on E. The image of the restriction $q|_{\widetilde{U}\times\mathbb{R}}$ is precisely $\varphi^{-1}(U)$. So we have a homeomorphism Φ_U given by a composition of homeomorphisms:

$$\varphi^{-1}(U) \xrightarrow{(q|_{\widetilde{U}\times\mathbb{R}})^{-1}} \widetilde{U} \times \mathbb{R} \xrightarrow{\varepsilon|_{\widetilde{U}} \times \mathbf{1}_{\mathbb{R}}} U \times \mathbb{R}, \qquad \Phi_U : \varphi^{-1}(U) \to U \times \mathbb{R} \tag{1.3}$$

It is straightforward to see that Φ_U is a vector space isomorphism on each fiber by construction. Moreover, for any $\overline{(x, y)} \in \varphi^{-1}(U)$, we have

$$\varphi\left(\overline{(x, y)}\right) = \varepsilon(x) = (\pi_U \circ \Phi_U)\left(\overline{(x, y)}\right). \tag{1.4}$$

Hence, Φ_U is a local trivialization. This establishes that φ is indeed a real vector bundle of rank 1 on S^1.

Now we show that this vector bundle is nontrivial. Suppose to the contrary there exists a global trivialization $\Phi_{S^1} : E \to S^1 \times \mathbb{R}$. If $\pi_{S^1} : S^1 \times \mathbb{R} \to S^1$ is projection onto the first factor then $\pi_{S^1} \circ \Phi_{S^1} = \varphi$, hence

$$(\pi_{S^1} \circ \Phi_{S^1})(\overline{(x, y)}) = \varphi\left(\overline{(x, y)}\right) = \varepsilon(x) = e^{2\pi ix}. \tag{1.5}$$

Therefore, we may write Φ_{S^1} as

$$\Phi_{S^1}\left(\overline{(x, y)}\right) = \left(e^{2\pi ix}, f\left(\overline{(x, y)}\right)\right) \tag{1.6}$$

for some function $f : E \to \mathbb{R}$.

For any equivalence class $\overline{(x, y)} \in E$, we say that the equivalence class is in *canonical form* if the equivalence class is written as $\overline{(x, y)}$ where $x \in [0, 1)$. Note that every element of E can be expressed uniquely in canonical form. From now on, we assume that our equivalence classes are written in canonical form unless stated or obviously otherwise.

Recall that for any $p \in S^1$, we have that the fiber $\pi^{-1}(\{p\}) = \bigcup_{y\in\mathbb{R}} \overline{(x_p, y)}$ where $\varepsilon(x_p) = p$. Since Φ_{S^1} is a vector space isomorphism with $\mathbb{R}$ on this fiber, it must act via $\overline{(x_p, y)} \mapsto \lambda_{x_p} y$ where $\lambda_{x_p} \in \mathbb{R} \setminus \{0\}$ for every $p \in S^1$. Let $\pi_{\mathbb{R}} \colon S^1 \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, and define $\Psi = \pi_{\mathbb{R}} \circ \Phi_{S^1} \circ q = f \circ q$. Ψ is a function $\mathbb{R}^2 \to \mathbb{R}$ and is continuous as a composition of three continuous functions. Hence, for all $y \in \mathbb{R}$,

$$\lim_{x\to 1^-} \Psi(x, y) = \Psi(1, y) = f\left(\overline{(1, y)}\right) = f\left(\overline{(0, -y)}\right) = -\lambda_0 y. \tag{1.7}$$

Hence,

$$\lim_{x\to 1^-} \lambda_x = \lim_{x\to 1^-} \Psi(x, 1) = -\lambda_0, \tag{1.8}$$

But now, since the map $x \mapsto \lambda_x$ (that is, the function $\Psi(x, 1)$) is continuous, the intermediate value theorem implies that there exists $\alpha \in [0, 1)$ such that $\lambda_\alpha = 0$. This is a contradiction since we required that $\lambda_x \in \mathbb{R} \setminus \{0\}$ for all $x \in [0, 1)$ so that Φ_{S^1} would be a vector space isomorphism on each fiber. Therefore, no global trivialization can exist.

Note that it took quite a bit of work to establish that the Möbius bundle is nontrivial. But hidden in the proof is a more powerful general idea that characterizes precisely when a vector bundle is nontrivial. To do this, we need to introduce more terminology.

Definition 2. *Let $\pi \colon E \to X$ be a vector bundle on X. A **section** of this vector bundle is a continuous map $s \colon X \to E$ such that $\pi \circ s$ is the identity map on X.*

In other words, a section is a choice of vectors over each point of the base space such that the vectors vary continuously as we move over the base space. Vector fields are a concrete example of this: they are simply the sections of the tangent bundle on a smooth manifold. It turns out that sections can detect when a vector bundle is trivial.

Proposition 1. *Let $\pi \colon E \to X$ be a vector bundle on X of rank r. The vector bundle is trivial if and only if there are r sections $s_1, s_2, \ldots, s_r$ such that $s_1(x), s_2(x), \ldots, s_r(x)$ are linearly independent for all $x \in X$.*

Proof. In one direction, suppose that $\pi \colon E \to X$ is trivial. Pick a standard basis $e_1, e_2, \ldots, e_r$ of $\mathbb{C}^r$. There is a global trivialization Φ_X that is an isomorphism between each fiber and $\mathbb{C}^r$. In particular, for each $x \in X$, we have that $\Phi_X|_{\pi^{-1}(\{x\})}$ pulls back the basis $e_1, \ldots, e_r$ to a basis $(\Phi_X|_{\pi^{-1}(\{x\})})^{-1}(e_1), (\Phi_X|_{\pi^{-1}(\{x\})})^{-1}(e_2), \ldots, (\Phi_X|_{\pi^{-1}(\{x\})})^{-1}(e_r)$ of the fiber $\pi^{-1}(\{x\})$. Moreover, it does this continuously as we vary x, and thus the $\Phi_X|_{\pi^{-1}(\{x\})}(e_i)$ are sections that are linearly independent for all $x \in X$.

In the other direction, suppose that there are r sections $s_1, \ldots, s_r$ such that $s_1(x), s_2(x), \ldots, s_r(x)$ are linearly independent for every $x \in X$. Define the map $\Phi \colon X \times \mathbb{C}^r \to E$ by $\Phi(x, z_1, z_2, \ldots, z_r) = \sum_{i=1}^{r} z_i s_i(x)$. This is clearly a vector space isomorphism to each fiber by construction and continuous since it is continuous after composition with a local trivialization and local trivializations are homeomorphisms. We will show that Φ^{-1} is also continuous, which will establish that Φ^{-1} is in fact a global trivialization.

Continuity is a local property so it suffices to work in a local trivialization $\Phi_U \colon \pi^{-1}(U) \to U \times \mathbb{C}^r$. We have the following commutative diagram.

$$\begin{array}{ccccc} U \times \mathbb{C}^r & \xrightarrow{\Phi|_{U\times\mathbb{C}^r}} & \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times \mathbb{C}^r \\ \downarrow & & \downarrow{\scriptstyle \pi|_{\pi^{-1}(U)}} & & \\ U & & U & & \end{array} \tag{1.9}$$

So we may identify $\Phi|_{U\times\mathbb{C}^r}$ with the composition $\Psi \coloneqq \Phi_U \circ \Phi|_{U\times\mathbb{C}^r}$, and since Φ_U is a homeomorphism and continuity is a local property, the continuity of Φ^{-1} is equivalent to that of Ψ^{-1}. By the above diagram,

$$\Psi(x, v) = \left(x, \Phi_U|_{\pi^{-1}(\{x\})} \left(\sum_{i=1}^{r} v_i s_i(x) \right) \right). \tag{1.10}$$

Let us abbreviate $T_x \coloneqq \Phi_U|_{\pi^{-1}(\{x\})}$. Then, T_x can be identified with an element of $\mathrm{GL}_n(\mathbb{C})$ for every x, since local trivializations are vector space isomorphisms on each fiber. Moreover, the map $x \mapsto T_x$ is a continuous map into the space $\mathrm{GL}_n(\mathbb{C})$ when it is identified with the space $\mathbb{C}^{n^2}$. The map $x \mapsto T_x^{-1}$ is also a continuous map into $\mathrm{GL}_n(\mathbb{C})$ since the matrix entries of T_x^{-1} are simply polynomials in the entries of T_x. Therefore, noting that $\Psi^{-1}(x, v) = (x, T_x^{-1}(v))$, we conclude that Ψ^{-1} is continuous. □

This result simplifies to something simple in the case of a vector bundle of rank 1, also called a *line bundle*. Proposition 1 implies that a line bundle is trivial if and only if there exists a nonvanishing section, since a single vector by itself forms a linearly independent set as long as it is not the zero vector. The argument we gave to show that the Möbius bundle on S^1 is nontrivial can be adapted to show that any section of the Möbius bundle must vanish, which implies the nontriviality of the bundle by Proposition 1. Geometrically, the fact that every section of the Möbius bundle must vanish corresponds to the fact that any continuous normal vector field on the Möbius strip must vanish somewhere. Proposition 1 also justifies how exactly the hairy ball theorem implies that S^2 has a nontrivial tangent bundle: since every continuous vector field on S^2 vanishes by the theorem, we can never find two (or even a single) linearly independent sections on S^2 and thus the tangent bundle of S^2 is nontrivial by Proposition 1.

1.2 Grassmannians and the Classification Problem

As simple as Proposition 1 is to state, showing that a vector bundle is or is not trivial is a hard problem in general. To illustrate the difficulty of the problem, we will show that for real vector bundles over a paracompact space, the question of triviality is equivalent to understanding whether a certain map from the space into a *classifying space* is nullhomotopic. Of course, classifying maps up to homotopy is a hard problem in general. To speak of the correspondence between these two problems, we need to define how to pull back a vector bundle via a continuous map.

Definition 3. *Let X be a topological space and let $p\colon E_1 \to X$ and $q\colon E_2 \to X$ be vector bundles over X. An **isomorphism** between p and q is a homeomorphism $\varphi\colon E_1 \to E_2$ such that φ is a vector space isomorphism between each fiber $p^{-1}(\{x\})$ and the fiber $q^{-1}(\{x\})$.*

It is clear from the definition that this notion of isomorphism is the correct one for vector bundles. In fact, there is a general notion of a morphism of vector bundles that allows us to form the category of vector bundles over some fixed space, which we will not need here.

Definition 4. *Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous map. Let $p\colon E \to Y$ be a vector bundle. Then there exists a vector bundle $p'\colon E' \to X$ and a map $f'\colon E' \to E$ such that for every $x \in X$, f' is a vector space isomorphism between the fiber of E' over x and the fiber of E over $f(x)$. Such a vector bundle E' is unique up to isomorphism and is called the **pullback** of E by f, often denoted $f^*(E)$.*

Proof. We define

$$E' \coloneqq \{(x, v) \in X \times E \colon f(x) = p(v)\}. \tag{1.11}$$

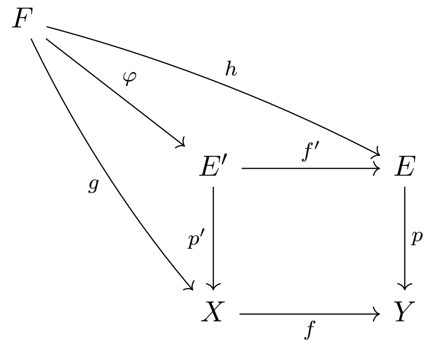
We also define the maps $f'\colon E' \to E$ and $p'\colon E' \to X$ by $f'(x, v) = v$ and $p'(x, v) = x$.

Let $\Gamma_f \subseteq X \times Y$ denote the graph of f. Note that the map $\mathbf{1} \times p\colon X \times E \to X \times Y$ defines a vector bundle over Γ_f. Indeed, for every $(x, f(x)) \in \Gamma_f$, the fiber $\{(x, v)\colon v \in p^{-1}(\{f(x)\})\}$ is clearly isomorphic to the fiber $p^{-1}(\{f(x)\})$. In fact, this same vector bundle arises even if we restrict $\mathbf{1} \times p$ to E', because for any $(x, f(x)) \in \Gamma_f$, we have

$$\begin{aligned}(\mathbf{1} \times p)^{-1}(\{(x, f(x))\}) &= \{(x, v)\colon v \in p^{-1}(\{f(x)\})\} \\ &= \{(x, v)\colon p(v) = f(x)\} \\ &= (\mathbf{1} \times p)|_{E'}^{-1}(\{(x, f(x))\}).\end{aligned} \tag{1.12}$$

Moreover, the projection map $\pi_1\colon \Gamma_f \to X$ given by $\pi_1(x, f(x)) = x$ is a homeomorphism when Γ_f is given the subspace topology with respect to the ambient space $X \times Y$. Therefore, the composition of $(\mathbf{1} \times p)|_{E'}$ with π_1 is itself a vector bundle over X. But this composition is exactly the map p', so p' is a vector bundle over X. The fiber over a fixed point $x_0 \in X$ in this vector bundle is $\{(x_0, v) \in X \times E \colon p(v) = f(x_0)\}$. The map f' takes this fiber and maps it to $\{v \in E\colon p(v) = f(x_0)\}$ which is precisely the fiber over $f(x_0)$.

To establish uniqueness, suppose that $g\colon F \to X$ is another vector bundle over X that satisfies the properties of E'. Then there exists a map $h\colon F \to E$ such that h is a vector space isomorphism between the fiber of F over x and the fiber of E over $f(x)$. Consider the map $\varphi\colon F \to E'$ given by $\varphi(v) = (g(v), h(v))$. Indeed, this lands in E' since $p \circ h = f \circ g$ by assumption. More generally, it can be checked that the following diagram commutes.



(1.13)

A simple check using the diagram above shows that φ maps fibers of F to corresponding fibers of E' through vector space isomorphisms. Now, an argument similar to the one at the end of Proposition 1 shows that φ must be an isomorphism of vector bundles. $\square$

Next, we construct the classifying space of real vector bundles on a paracompact space. Recall the Grassmannian $G(k, \mathbb{R}^n)$ where $0 \leq k \leq n$ is an integer, which is the space of all k-dimensional vector subspaces of $\mathbb{R}^n$. There are natural inclusions $\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \dots$, and so there are natural inclusions $G(k, \mathbb{R}) \subseteq G(k, \mathbb{R}^2) \subseteq \dots$. We can endow the Grassmannian with a topology. We define $V_k(\mathbb{R}^n)$ to be the *Stiefel manifold*, which is the space of k-tuples of orthonormal unit vectors in $\mathbb{R}^n$. Clearly, this Stiefel manifold is a topological space, as it can be identified with a subspace of the product $\prod_{i=1}^k S^{n-1}$. There exists a natural surjective map $V_k(\mathbb{R}^n) \to G(k, \mathbb{R}^n)$ that takes every k-tuple of orthonormal unit vectors and sends it to the span of those vectors. The topology on $G(k, \mathbb{R}^n)$ is defined to be the quotient topology with respect to this surjection.

Define

$$G(k, \mathbb{R}^\infty) := \bigcup_{n=1}^{\infty} G(k, \mathbb{R}^n). \tag{1.14}$$

We claim that this is the space of k-dimensional linear subspaces of the real vector space of countably infinite dimension (which we denote by $\mathbb{R}^\infty$). Of course, every element of $G(k, \mathbb{R}^\infty)$ can be represented as a k-dimensional linear subspace, since every element belongs to $G(k, \mathbb{R}^n)$ for some n and $\mathbb{R}^n$ embeds naturally inside $\mathbb{R}^\infty$. Conversely, given a k-dimensional linear subspace of $\mathbb{R}^\infty$, one can look at the ordering on the basis vectors of $\mathbb{R}^\infty$ and see that the subspace lives inside $\mathbb{R}^n$ for some sufficiently large n, and thus belongs to $G(k, \mathbb{R}^n) \subseteq G(k, \mathbb{R}^\infty)$.

The Grassmannian $G(k, \mathbb{R}^\infty)$ can be topologized with the direct limit topology coming from the direct system of inclusions $G(k, \mathbb{R}) \hookrightarrow G(k, \mathbb{R}^2) \hookrightarrow \dots$. More explicitly, $U \subseteq G(k, \mathbb{R}^\infty)$ is open if and only if $U \cap G(k, \mathbb{R}^n)$ is open for every $n \in \mathbb{N}$.

On each point of a Grassmannian $G(k, \mathbb{R}^n)$, one may naively attach the vector space described by the point itself. In fact, if we define $E_k(\mathbb{R}^n) := \{(\ell, v) \in G(k, \mathbb{R}^n) \times \mathbb{R}^n : v \in \ell\}$, then the projection onto the first factor $p\colon E_k(\mathbb{R}^n) \to G(k, \mathbb{R}^n)$ is a vector bundle. This bundle is often known as the *tautological bundle*. Showing that this is actually a vector bundle is not hard, though it is tedious and we will not do it here. We can extend this idea to $\mathbb{R}^\infty$ by defining $E_k(\mathbb{R}^\infty) = \bigcup_{n=1}^\infty E_k(\mathbb{R}^n)$ and endowing this set with the direct limit topology over the direct system of inclusions $E_k(\mathbb{R}) \hookrightarrow E_k(\mathbb{R}^2) \hookrightarrow \dots$. The tautological bundle in the case $n = \infty$ is indeed a vector bundle on $G(k, \mathbb{R}^\infty)$.

For a space X, let $\operatorname{Vect}^n(X)$ denote the real vector bundles of rank n on X up to isomorphism. We also use the standard notation of $[X, Y]$ to denote the set of homotopy classes of continuous maps $X \to Y$. We are now ready to state our main correspondence. This argument can be found in [10].

Theorem 1. *Let X be a paracompact topological space. $[X, G(n, \mathbb{R}^\infty)]$ is in a natural bijective correspondence with* $\operatorname{Vect}^n(X)$ *via $[f] \mapsto f^*(E_n(\mathbb{R}^\infty))$.*

Proof. Consider a continuous map $f\colon X \to G(n, \mathbb{R}^\infty)$. Suppose $\varphi\colon E \to f^*(E_n(\mathbb{R}^\infty))$ is an isomorphism of vector bundles. Then, the following diagram commutes

$$\begin{array}{ccccccc} E & \xrightarrow{\varphi} & f^*(E_n(\mathbb{R}^\infty)) & \xrightarrow{f'} & E_n(\mathbb{R}^\infty) & \xrightarrow{\pi} & \mathbb{R}^\infty \\ & \searrow^{p} & \downarrow & & \downarrow & & \\ & & X & \xrightarrow[f]{} & G(n, \mathbb{R}^\infty) & & \end{array} \tag{1.15}$$

where $\pi(\ell, v) = v$ is projection onto the second factor and the map f' is the map induced by the pullback. Note that the maps φ, f', and π are all vector space isomorphisms on fibers by definition, hence their composition $\pi \circ f' \circ \varphi\colon E \to \mathbb{R}^\infty$ is a vector space isomorphism on fibers. In particular, the composition is a linear injection on each fiber. So every continuous map $f\colon X \to G(n, \mathbb{R}^\infty)$ determines a linear injection on fibers $E \to \mathbb{R}^\infty$ for any E isomorphic to $f^*(E_n(\mathbb{R}^\infty))$.

Conversely, suppose we are given a linear injection on fibers $g\colon E \to \mathbb{R}^\infty$ where $p\colon E \to X$ is a real vector bundle of rank n. We define the map $f\colon X \to G(n, \mathbb{R}^\infty)$ by $f(x) = g(p^{-1}(\{x\}))$. Note that $g(p^{-1}(\{x\}))$ is the image of a fiber under a linear injection of fibers, and is thus itself a vector subspace of $\mathbb{R}^\infty$ of dimension $\dim p^{-1}(\{x\}) = n$. So f is indeed a well-defined map into $G(n, \mathbb{R}^\infty)$. It is clearly continuous since g is. Moreover, unravelling the definitions, we can see that $f^*(E_n(\mathbb{R}^\infty)) \subseteq X \times G(n, \mathbb{R}^\infty) \times \mathbb{R}^\infty$. More precisely,

$$f^*(E_n(\mathbb{R}^\infty)) = \{(x, f(x), w) \in X \times G(n, \mathbb{R}^\infty) \times \mathbb{R}^\infty \colon w \in f(x)\}. \tag{1.16}$$

Define the map $\varphi\colon E \to f^*(E_n(\mathbb{R}^\infty))$ by $\varphi(x, v) = (x, f(x), g(x, v))$. It is clear that this choice of φ is an isomorphism. Hence, every linear injection on fibers $E \to \mathbb{R}^\infty$ determines a continuous map $f\colon X \to G(n, \mathbb{R}^\infty)$ and an isomorphism $\varphi\colon E \to f^*(E_n(\mathbb{R}^\infty))$. It is not hard to see that our constructions are inverses of each other, so we have established a correspondence between continuous maps $f\colon X \to G(n, \mathbb{R}^\infty)$ and the linear injections of the form $E \to \mathbb{R}^\infty$ for E isomorphic to the pullback bundle $f^*(E_n(\mathbb{R}^\infty))$.

Let $\Psi\colon [X, G(n, \mathbb{R}^\infty)] \to \operatorname{Vect}^n(X)$ be the map $\Psi([f]) = f^*(E_n(\mathbb{R}^\infty))$. Showing that this map is well-defined is tantamount to showing that pullbacks of vector bundles are homotopy invariant.

This is a technical point that relies on the paracompactness of the base space X, which is not found in [10]. One may consult [12] and [13] for the details.

Assuming now that Ψ is well-defined, we will show that it is injective. Let E be isomorphic to $f_1^*(E_n(\mathbb{R}^\infty))$ and $f_2^*(E_n(\mathbb{R}^\infty))$ where f_1 and f_2 are continuous maps $X \to G(n, \mathbb{R}^\infty)$. By our correspondence above, these determine linear injections on fibers $E \to \mathbb{R}^\infty$ which we will call g_1 and g_2, respectively. Define the maps $H_1, H_2 \colon \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ (where I is the unit interval) via

$$H_1(x_1, x_2, \ldots, t) = (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0, \ldots) \tag{1.17}$$

$$H_2(x_1, x_2, \ldots, t) = (1-t)(x_1, x_2, \ldots) + t(0, x_1, 0, x_2, \ldots). \tag{1.18}$$

For every fixed t, each of these define a linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ with trivial kernel. Define the functions $g_1', g_2' \colon E \to \mathbb{R}^\infty$ by $g_1' = H_1|_{\mathbb{R}^\infty \times \{1\}} \circ g_1$ and $g_2' = H_2|_{\mathbb{R}^\infty \times \{1\}} \circ g_2$. Clearly, g_1 and g_1' are homotopic and g_2 and g_2' are homotopic. Finally, define $H \colon E \times I \to \mathbb{R}^\infty$ by

$$H(s, t) = (1-t)g_1'(s) + t g_2'(s). \tag{1.19}$$

This is a homotopy between g_1' and g_2'. We conclude that g_1 and g_2 are homotopic. By construction, for every fixed t this homotopy is a linear injection on each fiber in the variable s. Now, we can note that $H(p^{-1}(\{x\}), t)$ defines a homotopy from f_1 to f_2. That is, $[f_1] = [f_2]$ and we have established that Ψ is injective.

Finally, we show that Ψ is surjective. Let $p \colon E \to X$ be a n-dimensional real vector bundle. By paracompactness, there exists a countable open cover $\{U_i\}_{i \in \mathbb{N}}$ of X such that E is trivial over each U_i and there exists a partition of unity $\{\varphi_i\}_{i \in \mathbb{N}}$ that is subordinate to the open cover $\{U_i\}_{i \in \mathbb{N}}$. We denote the local trivialization of E over U_i by $\Phi_i \colon p^{-1}(U_i) \to U_i \times \mathbb{R}^n$. Define $g_i \colon p^{-1}(U_i) \to \mathbb{R}^n$ by the composition of Φ_i with the projection onto the $\mathbb{R}^n$ factor. Put

$$h_i(s) = \begin{cases} \varphi_i(p(s)) g_i(s) & v \in p^{-1}(U_i) \\ (0, 0, \ldots, 0) & \text{otherwise} \end{cases} \tag{1.20}$$

These maps $h_i \colon E \to \mathbb{R}^n$ are continuous and linear by construction. Only finitely many of the h_i fail to vanish at any given point in E, hence the function $h \colon E \to \mathbb{R}^\infty$ given by $h(s) = (h_1(s), h_2(s), \ldots)$ is a well-defined linear map into $\mathbb{R}^\infty$. Finally, observe that the kernel of this map is trivial because for any $s \in E$, as long as this point is not of the form $(x, 0)$ in local coordinates, at least one of the h_i will fail to vanish at s. Hence, we have determined a linear injection on fibers $h \colon E \to \mathbb{R}^\infty$. By our work above, this corresponds to some continuous map $f \colon X \to G(n, \mathbb{R}^\infty)$ and where E is isomorphic to $f^*(E_n(\mathbb{R}^\infty))$. This establishes that Ψ is surjective. □

Theorem 1 holds in the case of complex vector bundles as well. In that case, the classifying space is the Grassmannian $G(n, \mathbb{C}^\infty)$.

Theorem 1 shows us that understanding the isomorphism classes of vector bundles on a paracompact space is equivalent to understanding the homotopy classes of continuous maps $X \to G(n, \mathbb{R}^\infty)$. In other words, maps into $G(n, \mathbb{R}^\infty)$ classify vector bundles over a paracompact space. For this reason, we say that $G(n, \mathbb{R}^\infty)$ is the *classifying space* of real vector bundles. For any real vector bundle over X, any map $X \to G(n, \mathbb{R}^\infty)$ in the corresponding homotopy class is called the *classifying map* of the vector bundle.

In general, understanding the space $[X, G(n, \mathbb{R}^\infty)]$ is impossible. The proof of Theorem 1 implies that a vector bundle is trivial precisely when its classifying map is nullhomotopic, and this is not something we can easily determine in general. However, nullhomotopic maps induce trivial homomorphisms in cohomology (though the converse is not true) and it is often easier to detect whether an induced homomorphism is trivial. Topologically this is what *Chern classes* detect, and we will study them in Chapter 3 in the algebro-geometric context.

1.3 Locally Free Sheaves and Projectivization

We will now adapt vector bundles to the setting of algebraic geometry. For the rest of the discussion, unless stated otherwise, we will assume that our base field for our schemes is $\mathbb{C}$ and that all of our vector bundles are complex.

One of the basic pieces of data in a scheme is its structure sheaf. Sheaves are the canonical tools we use for keeping track of local data on a topological space. It is therefore unsurprising that certain sheaves can effectively play the role of vector bundles.

Definition 5. *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called* ***locally free of rank*** r *if there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that $\mathcal{F}|_{U_\alpha} \cong \bigoplus_{i=1}^r \mathcal{O}_X(U_\alpha)$ for every $\alpha \in A$.*

Locally free sheaves are exactly the type of sheaves that play the role of vector bundles. We make this precise with the following result.

Theorem 2. *Let X be a scheme. There is a natural bijective correspondence between the locally free sheaves of rank r on X and vector bundles of rank r on X.*

Proof. On one hand, let $p \colon E \to X$ be a vector bundle of rank r. We define a sheaf $\mathcal{F}$ on X. In particular, define

$$\mathcal{F}(U) := \{s \colon U \to E \colon s \text{ is a } \mathbb{C}\text{-morphism section of } E \text{ over } U\}, \tag{1.21}$$

for every open subset $U \subseteq X$. In the above definition, we mean "section" in the sense of Definition 2. It is straightforward to check that the above actually defines a sheaf. In fact, this is a sheaf of $\mathcal{O}_X$-modules: for every open set $U \subseteq X$ and $s \in \mathcal{F}(U)$, we can multiply s with $\varphi \in \mathcal{O}_X(U)$ by just pointwise scalar multiplication.

Suppose p is trivial over the open subset $V \subseteq X$. Then E restricted to this subset is isomorphic to $U \times \mathbb{A}^r$. Then, we have that

$$\mathcal{F}(V) \cong \{s \colon V \to \mathbb{A}^r \colon s \text{ is a } \mathbb{C}\text{-morphism}\}. \tag{1.22}$$

But of course, a $s\colon V \to \mathbb{A}^r$ is uniquely determined by r regular functions from $\mathcal{O}_V$. Hence, $\mathcal{F}|_V \cong \bigoplus_{i=1}^r \mathcal{O}_X(V)$. Hence, $\mathcal{F}$ is a locally free sheaf.

On the other hand, suppose we are given a locally free sheaf $\mathcal{F}$ over X. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X such that there exist isomorphisms $\Psi_\alpha\colon \mathcal{F}|_{U_\alpha} \to \bigoplus_{i=1}^r \mathcal{O}_X(U_\alpha)$. For $\alpha, \beta \in A$, we can restrict Ψ_α to $U_\alpha \cap U_\beta$ to obtain the isomorphism $\Psi_\alpha|_{U_\alpha\cap U_\beta}\colon \mathcal{F}|_{U_\alpha\cap U_\beta} \to \bigoplus_{i=1}^r \mathcal{O}_X(U_\alpha \cap U_\beta)$. Similarly, we can restrict Ψ_β to $U_\alpha \cap U_\beta$. Then, we may consider the composition

$$\Psi_\alpha|_{U_\alpha\cap U_\beta} \circ \Psi_\beta|_{U_\alpha\cap U_\beta}^{-1}\colon \bigoplus_{i=1}^r \mathcal{O}_X(U_\alpha \cap U_\beta) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(U_\alpha \cap U_\beta). \tag{1.23}$$

For brevity, we denote the above map by $\Psi_{\alpha,\beta}$. This is an automorphism which can be represented by an $r \times r$ matrix of regular functions on $U_\alpha \cap U_\beta$.

Now we glue the schemes $U_\alpha \times \mathbb{A}^r$ and $U_\beta \times \mathbb{A}^r$ along the intersection $(U_\alpha \cap U_\beta) \times \mathbb{A}^r$ using the isomorphism $(U_\alpha \cap U_\beta) \times \mathbb{A}^r \to (U_\alpha \cap U_\beta) \times \mathbb{A}^r$ given by $(x, v) \mapsto (x, \Psi_{\alpha,\beta}(v))$

Upon finishing this gluing, we have obtained a rank r vector bundle over X. This construction is exactly the opposite of our previous construction, hence we have established a correspondence between vector bundles and locally free sheaves. □

Hence, from a scheme-theoretic point of view, locally free sheaves carry exactly the same data as vector bundles. We can even single out the fibers. If $p\colon E \to X$ is a vector bundle over a scheme X, the fiber over the point $x \in X$ is simply the preimage $p^{-1}(\{x\})$ which has a natural vector space structure. Alternatively, we may consider the inclusion $i\colon \{x\} \hookrightarrow X$. Then the fiber over x is encoded in the pullback bundle $i^*(E)$. This is a vector bundle over a singleton and is thus just the data of the vector space sitting over the point x in the original vector bundle E. Analogously, if $\mathcal{F}$ is the locally free sheaf on X corresponding to p, then the fiber over $x \in X$ is the pullback sheaf $i^*\mathcal{F}$. Since this sheaf is a sheaf over a single point, there is only one piece of data carried and that is the collection of global sections $(i^*)(\mathcal{F})(\{x\})$, which is exactly the vector space sitting on top of the point x by the construction in Theorem 2.

Recall also that the pullback of a locally free sheaf is locally free. This suggests that the pullback of locally free sheaves is in correspondence with the pullback of vector bundles in the sense of Definition 4. This needs checking, but is true (see [2]). The geometric construction of the direct sum of vector bundles, which we have no explicitly described here, easily corresponds to the direct sum of locally free sheaves. There are also geometric definitions for the tensor products and duals of vector bundles. We will be more interested in the sheaf-theoretic versions of these operations. For the geometric definitions, one may consult [10].

In algebraic geometry, it is often the case that the theory is simplified when one replaces affine space with projective space. In the constructions we have provided so far, the fibers of a vector bundle are just copies of complex affine space. One may consider what happens when the affine fibers are replaced with projective ones, while the manner of gluing the fibers on the space remains unchanged. The new object that results from replacing fibers with their projectivizations without changing the manner of gluing is called the *projectivization* of the original vector bundle.

More precisely, let $p\colon E \to X$ be a vector bundle on a scheme X. Note that in the proof of Theorem 2 we saw that the geometric manner by which fibers of $p\colon E \to X$ are glued to the base space X is encoded by transition maps $\Psi_{\alpha,\beta}$. This is not unlike the way that the transition maps of a smooth manifold determine how the Euclidean patches are glued together. In particular, these transition maps induced automorphisms of the schemes $(U_\alpha \cap U_\beta) \times \mathbb{A}^r$ given by $(x,v) \mapsto (x, \Psi_{\alpha,\beta}(v))$ and those automorphisms then told us how the patches $U_\alpha \times \mathbb{A}^r$ and $U_\beta \times \mathbb{A}^r$ are glued together.

The transition maps $\Psi_{\alpha,\beta}$ are linear maps $\mathbb{A}^r \to \mathbb{A}^r$. In particular, they respect scalar multiplication. This means that the transition maps descend to maps $\mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$. Therefore, we may define the projectivization of the vector bundle $p\colon E \to X$, denoted $\tilde{p}\colon \mathbb{P}(E) \to X$, to be the object resulting from gluing the sets $U_\alpha \times \mathbb{P}^{r-1}$ and $U_\beta \times \mathbb{P}^{r-1}$ via the automorphisms $(x,v) \mapsto (x, \Psi_{\alpha,\beta}(v))$. It is clear that this construction gives an "projective bundle of rank $r-1$" where each fiber is now projective instead of affine, but still glued onto the base space in the same manner as the fibers of the original vector bundle.

The space $\mathbb{P}(E)$ also inherits a natural scheme structure through a Proj construction, which we will not describe here. We will invoke this fact later in Proposition 3. For the full details of this construction, one may consult [9].

At this point, we have a vector bundle on X and a continuous map $\tilde{p}\colon \mathbb{P}(E) \to X$. Consider the pullback of E by $\tilde{p}$. This gives us the diagram

$$\begin{array}{ccc} \tilde{p}^*(E) & & E \\ \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} \\ \mathbb{P}(E) & \xrightarrow[\tilde{p}]{} & X \end{array} \tag{1.24}$$

Unravelling Definition 4, we see that

$$\tilde{p}^*(E) = \{((x,v),w) \in \mathbb{P}(E) \times E \colon p(w) = x\} = \{((x,v),w) \colon v \in \tilde{p}^{-1}(\{x\}), w \in p^{-1}(\{x\})\}, \tag{1.25}$$

and $p'\colon p^*(E) \to \mathbb{P}(E)$ acts via $((x,v),w) \mapsto (x,v)$. In the vector bundle $p'\colon \tilde{p}^*(E) \to \mathbb{P}(E)$, there exists a subbundle $S \subseteq \tilde{p}^*(E)$ cut out by the equations $v_i w_j = v_j w_i$ for all $i,j \in \{1,2,\ldots,r\}$. Fixing a point $(x_0, v_0) \in \mathbb{P}(E)$, we can see that the fiber of S over this point is precisely the collection of points $((x_0,v_0),w) \in \mathbb{P}(E) \times E$ such that $w \in p^{-1}(\{x_0\})$ and v_0 is the projective equivalence class of w. It is clear that this fiber is one-dimensional, so S is a line bundle on $\mathbb{P}(E)$. This line bundle is called the *tautological subbundle* on $\mathbb{P}(E)$.

In fact, we can construct a whole class of line bundles on $\mathbb{P}(E)$. For every integer d, we may construct a sheaf $\mathcal{O}_{\mathbb{P}(E)}(d)$. We model these sheaves locally using the twisted sheaves $\mathcal{O}_{\mathbb{P}^{r-1}}(d)$. For every integer d, this is the sheaf whose sections are locally of the form f/g where f and g are regular functions with $\deg f - \deg g = d$. One may show that the twisted sheaves are locally isomorphic to the structure sheaf (see [7]), and thus they are locally free sheaves of rank 1 or line bundles.

To construct $\mathcal{O}_{\mathbb{P}(E)}(d)$, we note that the vector bundle $\mathbb{P}(E)$ is locally trivial over the sets U_α and is thus isomorphic to $U_\alpha \times \mathbb{P}^{r-1}$ over those sets. On each of these local trivializations, we can easily extend the sheaf $\mathcal{O}_{\mathbb{P}^{r-1}}(d)$. On the overlaps $(U_\alpha \cap U_\beta) \times \mathbb{P}^{r-1}$, we can glue using the transition maps $\Psi_{\alpha,\beta}$. In particular, we can identify $\varphi \in \mathcal{O}_{\mathbb{P}(E)}(d)(U_\alpha)$ with $\varphi \circ \Psi_{\alpha,\beta} \in \mathcal{O}_{\mathbb{P}(E)}(d)(U_\beta)$.

With this construction, one may identify tautological subbundle S with the line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$. Let $\mathscr{S}$ be the sheaf corresponding to the line bundle S in the sense of Theorem 2. Over open subsets $U \subseteq \mathbb{P}(E)$, we may define the morphism $\mathcal{O}_{\mathbb{P}(E)}(-1)(U) \to \mathscr{S}(U)$ by $\varphi \mapsto [(x, v) \mapsto ((x, v), \varphi v)]$. Doing this over all open subsets U gives a sheaf isomorphism $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathscr{S}$.

These constructions will be useful to us when we connect our theory of Chern classes with the enumerative problem of counting lines on a cubic surface.

Chapter 2

Chow Rings

In this chapter, we work to partially set up *Chow rings*. These objects play a role similar to that of singular cohomology rings in algebraic topology, especially in the context of what we will use them for later in Chapter 3. In general, they are good invariants of a scheme.

We first define the group structure of the Chow rings. We will then define a special case of the multiplication operation, and this will be sufficient for the type of enumerative problem we are interested in.

2.1 Divisors on Curves

We will first study the simplest example of a Chow group. This will motivate the general definition of Chow groups, similar to how the fundamental group motivates the definition of the higher homotopy groups in topology.

Definition 6. *Let $C \subseteq \mathbb{P}^n$ be a smooth irreducible projective curve. A* ***divisor*** *on C is a formal finite linear combination $D = \sum_{i=1}^{m} a_i P_i$ where $P_i \in C$ and $a_i \in \mathbb{Z}$ for all i. The* ***degree*** *of D is defined to be $\sum_{i=1}^{m} a_i$.*

The divisors on a curve have a natural group structure: every two divisors can be added as a sum of two formal linear combinations. The collection of all divisors on a curve C is denoted $\operatorname{Div} C$. This is the free abelian group generated by all the points of our curve. As such, it is quite a large group that does not record any of the structure intrinsic to the curve and so there is nothing very interesting to say about this group. However, we will argue that a certain quotient of this group is a far more interesting object.

Consider a zero-dimensional projective subscheme X of $\mathbb{P}^n$. One can show that this subscheme is affine and so is of the form $\operatorname{Spec} R$ for some $\mathbb{C}$-algebra R. Since this subscheme is zero-dimensional and Noetherian, it can be interpreted as corresponding to a finite set of points in $\mathbb{P}^n$ (c.f. source), though intrinsic to the scheme structure is the fact that these points are being recorded with certain *multiplicities*, which is a notion we make more precise now.

Without loss of generality, we may assume that X corresponds to a single point in $\mathbb{P}^n$, and after a change of coordinates, we can assume that this point is the origin in $\mathbb{A}^n \subseteq \mathbb{P}^n$. Then R is the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]/J$, where J is the ideal of the origin. Note that each coordinate function x_i vanishes at the origin, so $\langle x_1, \ldots, x_n \rangle \subseteq I(Z(J)) = \sqrt{J}$, with the last equality following from the Nullstellensatz. This means that for every $1 \leq i \leq n$, there exist positive integers d_i such that $x_i^{d_i} \in J$. Letting $d = \max_{1 \leq i \leq n} d_i$, we see that $x_i^d \in J$ for all d because J is an ideal. For a monomial in $\mathbb{C}[x_1, \ldots, x_n]$ with degree at least dn, the pigeonhole principle requires that there exists some $1 \leq i \leq n$ such that x_i^d divides that monomial, and since $x_i^d \in J$ and J is an ideal, it follows that the monomial is in J. That is, J contains all monomials with degree at least dn, so $R = \mathbb{C}[x_1, \ldots, x_n]/J$ must contain a basis of polynomials of degree less than dn when viewed as a complex vector space. In particular, R is a finite-dimensional $\mathbb{C}$-vector space.

Hence, for any zero-dimensional projective subscheme $X = \operatorname{Spec} R$, R is a finite-dimensional complex vector space. The dimension $\dim_{\mathbb{C}} R$ is called the *length* of the scheme X. We interpret the length as recording the number of points in the scheme X *counted with multiplicity.* Therefore, we can associate a divisor $a_1 P_1 + \cdots + a_m P_m$ to the zero-dimensional subscheme X, where the P_i are exactly the points in X and the a_i are the multiplicities of the points P_i.

Now suppose $C \subseteq \mathbb{P}^n$ be a smooth irreducible projective curve. Consider a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$. Let us further assume that f is not zero in the homogeneous coordinate ring $S(C)$ of C. This asssumption is equivalent to assuming that C is not contained in the vanishing set $Z(f)$. In this case, we can interpret $C \cap Z(f)$ as a zero-dimensional projective subscheme of $\mathbb{P}^n$. We denote the divisor associated to this subscheme $(f)_C$ or simply (f) when the curve C is clear from context. We are now ready to make a definition.

Definition 7. *Let C be a smooth irreducible projective curve. Suppose that $\varphi \in K(C)$ is a nonzero rational function on C. Writing $\varphi = f/g$ for $f, g \in S(C)^{(d)}$, we define the* ***divisor of the rational function*** *φ to be $(\varphi) := (f) + (g)$. This divisor is well-defined.*

Proof. There are two aspects of well-definedness to check. First note that we have defined divisors associated to homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$, but we have not shown that this association descends to a well-defined association for elements of the homogeneous coordinate ring $S(C)$. That is, if f and g are polynomials that have the same class $[f] = [g] \in S(C)$, we must show that $(f) = (g)$. Indeed, if $[f] = [g]$, then $g = f + h$ for some $h \in I(C)$. But since h vanishes on all of C, we have that $C \cap Z(f) = C \cap Z(f + h) = C \cap Z(g)$, and thus $(f) = (g)$ as desired.

We must also check that our definition of (φ) is invariant with respect to the choice of f and g chosen from $S(C)^{(d)}$ to represent $\varphi = \frac{f}{g}$. To do this, we will establish that if F and G are nonzero homogeneous elements of the coordinate ring $S(C)$, then $(FG) = (F) + (G)$.

Let us write $(FG) = \sum_{i=1}^m a_i P_i$. Of course, the the zero set of FG is simply the union of the zero sets of F and G. Hence, we can write $(F) = \sum_{i=1}^m b_i P_i$ and $(G) = \sum_{i=1}^m c_i P_i$, where some of the b_i and c_i might be zero. Fix $1 \leq j \leq n$ and consider an affine open neighborhood of P_j, say $\operatorname{Spec} R$. Suppose this neighborhood is small enough that it does not contain P_i for any $i \neq j$ (such an affine neighborhood exists because $C \cap Z(FG)$ is a zero-dimensional scheme and thus discrete).

By definition, we have $a_j = \dim_{\mathbb{C}} R/\langle FG\rangle$, $b_j = \dim_{\mathbb{C}} R/\langle F\rangle$, and $c_j = \dim_{\mathbb{C}} R/\langle G\rangle$. But of course, we have the short exact sequence

$$0 \longrightarrow {}^{R}\!/\!_{\langle F\rangle} \xrightarrow{\cdot G} {}^{R}\!/\!_{\langle FG\rangle} \xrightarrow{\cdot 1} {}^{R}\!/\!_{\langle G\rangle} \longrightarrow 0 \tag{2.1}$$

so by the additivity of dimension on exact sequences of vector spaces, we have that $a_j = b_j + c_j$. Repeating this argument for all j, we have shown that $(FG) = (F) + (G)$ as claimed.

Hence, if φ has another representation as $\varphi = f'/g'$, then we will have $f/g = f'/g'$ or $fg' = f'g$, which implies that $(f) + (g') = (f') + (g)$ or $(f) - (g) = (f') - (g')$ as desired. $\square$

Note that the divisor of any nonzero rational function on a projective curve has degree zero. If $\varphi \in K(C)$ is nonzero with $\varphi = \frac{f}{g}$, then note that $\deg(f)$ is the degree of the divisor associated to the intersection scheme $C \cap Z(f)$, which is $\deg f \deg C$ by Bézout's theorem. Similarly, $\deg(g) = \deg g \deg C$. But $\deg f = \deg g$ by the definition of rational functions on a projective curve. Therefore, $\deg(\varphi) = \deg(f) - \deg(g) = 0$.

Note also that since we have shown that $(FG) = (F) + (G)$ for nonzero homogeneous elements F and G in the coordinate ring $S(C)$, the divisors of the form (φ) for $\varphi \in K(C) \setminus \{0\}$ form a subgroup of $\operatorname{Div} C$. More precisely, this is because $(\varphi) + (\psi) = (\varphi\psi)$ for $\varphi, \psi \in K(C) \setminus \{0\}$. The quotient of $\operatorname{Div} C$ by this subgroup is of interest.

Definition 8. *Let C be a smooth irreducible projective curve. The quotient* $\operatorname{Div} C$ *by the subgroup of divisors of the form (φ) for $\varphi \in K(C) \setminus \{0\}$ is called the **Picard group**, denoted by* $\operatorname{Pic} C$*. If two divisors in* $\operatorname{Div} C$ *determine the same class in the quotient* $\operatorname{Pic} C$*, they are said to be **linearly equivalent**.*

The important assumption that makes the Picard group interesting is the assumption that we are dealing with *projective* curves. In this case, our rational functions are forced to be given by ratios of homogeneous elements from the coordinate ring of the same degree. This is a severe restriction on the space of rational functions on a projective curve, and so the process of quotienting $\operatorname{Div} C$ by divisors arising from rational functions does not kill off too many divisors thus leaving behind a quotient group that is rich enough to be a useful invariant. Moreover, the Picard group of a curve is often computable which makes it a practical tool for telling curves apart. A simple example is the case of $\mathbb{P}^1$, where one may show that all divisors of the same degree on the projective line are linearly equivalent, so $\operatorname{Pic} \mathbb{P}^1 \cong \mathbb{Z}$.

The upshot of this discussion is that the development of the Picard group will completely parallel our development of the Chow groups. In fact, the zeroth Chow group on a smooth projective curve is exactly the Picard group of that curve. The Picard group was formed by taking formal linear combinations of zero-dimensional subvarieties of a curve and quotienting out by a notion of "rational" (read: linear) equivalence. This is somewhat similar in spirit to the idea behind singular homology, where one takes an enormous free abelian group on all singular simplices of a certain dimension and quotients by an equivalence relation induced by a boundary map. Different homology groups correspond to quotients of free abelian groups on simplices of different dimension, which suggests the direction in which we ought to generalize the construction of the Picard group. Indeed, we will adapt the construction to formal linear combinations of higher dimensional subvarieties of an arbitrary scheme, thus defining the Chow groups.

2.2 Chow Groups

The real challenge of defining the Picard group was in defining the divisor associated to a rational function on a curve. We gave most of the details of this construction in Section 2.1. Defining the divisor associated to a rational function on a higher-dimensional scheme is more involved and requires some technical notions from commutative algebra. The standard references for this construction are quite scattered and leave out many details, so we work to provide them here. One may consult [7], [9], and [11].

Let R be a Noetherian ring and let M be a finitely-generated R-module. One can show that M admits a finite filtration by submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M, \tag{2.2}$$

such that for each $1 \leq i \leq r$, we have that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i \subseteq R$ (see [11]). The construction of such a filtration is fairly straightforward. We begin by defining M_1 such that $M_1 \cong M/\mathfrak{p}_1$ where $\mathfrak{p}_1$ is an associated prime ideal of M. We repeat this procedure, constructing submodules of M/M_{i-1} by picking from associated primes. By Noetherianity, this process must terminate, giving us a finite chain as in 2.2.

Such a filtration is called a *prime filtration.* Note that this is not quite the same as a *composition series* in the sense of commutative algebra. In a composition series, we merely require each module in the filtration to be a simple module, which is equivalent to requiring that successive quotients in the filtration are of the form $R/\mathfrak{m}_\mathfrak{i}$ for some *maximal* ideals $\mathfrak{m}_i \subseteq R$.

The prime filtration (or even composition series) may not be unique, however the amount of times a given minimal prime ideal $\mathfrak{p} \subseteq R$ appears in a successive quotient is independent of the prime filtration of the module M chosen. This can be seen by localization. Consider a prime filtration of M as in 2.2 and let $\mathfrak{p} \subseteq R$ be a minimal prime ideal. We may localize the filtration at the prime ideal $\mathfrak{p}$ to obtain

$$0 = (M_0)_\mathfrak{p} \subseteq (M_1)_\mathfrak{p} \subseteq \cdots \subseteq (M_r)_\mathfrak{p} = M_\mathfrak{p}, \tag{2.3}$$

Taking successive quotients, observe that since localization commutes with quotients, we have

$$(M_i)_\mathfrak{p} \big/ (M_{i-1})_\mathfrak{p} \cong \left(M_i \big/ M_{i-1} \right)_\mathfrak{p} \cong \left(R \big/ \mathfrak{p}_i \right)_\mathfrak{p} \cong R_\mathfrak{p} \big/ \mathfrak{p}_i R_\mathfrak{p} \tag{2.4}$$

If $\mathfrak{p} = \mathfrak{p}_i$, then observe that $R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field, namely the fraction field of $R/\mathfrak{p}$, since

$\mathfrak{p}R_\mathfrak{p}$ is the maximal ideal of the local ring $R_\mathfrak{p}$. On the other hand, suppose $\mathfrak{p} \neq \mathfrak{p}_i$. Note that $\mathfrak{p}_i R_\mathfrak{p}$ is either a prime ideal of $R_\mathfrak{p}$ or all of $R_\mathfrak{p}$ itself because $\mathfrak{p}_i$ is a prime ideal of R. But the prime ideals of $R_\mathfrak{p}$ are in natural correspondence with the prime ideals of R contained in $\mathfrak{p}$. Since $\mathfrak{p}$ is a minimal prime ideal and $\mathfrak{p}_i \neq \mathfrak{p}$, it must be the case that $\mathfrak{p}_i \nsubseteq \mathfrak{p}$. In other words, $\mathfrak{p}_i R_\mathfrak{p}$ corresponds to the prime ideal $\mathfrak{p}_i$ which is *not* contained in $\mathfrak{p}$. Hence, we must have $\mathfrak{p}_i R_\mathfrak{p} = R_\mathfrak{p}$ and thus $R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p} \cong 0$ if $\mathfrak{p} \neq \mathfrak{p}_i$.

So by 2.4, we have established that

$$(M_i)_\mathfrak{p} \big/ (M_{i-1})_\mathfrak{p} \cong \begin{cases} R_\mathfrak{p} \big/ \mathfrak{p}R_\mathfrak{p} & \text{if } \mathfrak{p} = \mathfrak{p}_i \\ 0 & \text{if } \mathfrak{p} \neq \mathfrak{p}_i. \end{cases} \tag{2.5}$$

This tells us that the chain 2.3 becomes a composition series after possibly removing repeating submodules, because once we remove the repeating submodules, successive quotients are isomorphic to $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $\mathfrak{p}R_\mathfrak{p}$ is a maximal ideal of $R_\mathfrak{p}$. This is the composition series of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ and by 2.5, the length of this composition series is exactly the number of times $\mathfrak{p}$ appears in the prime filtration 2.2. Hence, the number of times $\mathfrak{p}$ appears in the prime filtration 2.2 is exactly the length of the module $M_\mathfrak{p}$ which only depends on M and $\mathfrak{p}$.

We will now adapt this argument to the case we need. We will strengthen our assumptions on R and M and show that under these new assumptions, not only is our result true for minimal prime ideals, but also height 1 prime ideals. Let us suppose R is a Noetherian domain and $M = R/\langle f \rangle$ where $f \in R$ is nonzero. In this case, suppose that $\mathfrak{p}$ is a height 1 prime ideal of R. We claim that the number of times $\mathfrak{p}$ appears in a prime filtration of M is independent of the filtration chosen.

One possibility is that $\langle f \rangle \nsubseteq \mathfrak{p}$. In such a case, there is an element of $\langle f \rangle$ outside $\mathfrak{p}$, hence the localization $M_\mathfrak{p} = (R/\langle f \rangle)_\mathfrak{p}$ is localization with respect to a multiplicatively closed subset that contains the coset of an element of $\langle f \rangle$, which is the zero element in the quotient. Hence, $M_\mathfrak{p} \cong 0$. Therefore, submodules of $M_\mathfrak{p}$ are all isomorphic to 0 and it follows that $(M_i/M_{i-1})_\mathfrak{p} \cong (M_i)_\mathfrak{p}/(M_{i-1})_\mathfrak{p} \cong 0$ for every i. In particular, $M_i/M_{i-1} \ncong R/\mathfrak{p}$ for any i, since the localization of $R/\mathfrak{p}$ at $\mathfrak{p}$ is the fraction field $\operatorname{Frac}(R/\mathfrak{p})$ which is not isomorphic to 0. In other words, if $\langle f \rangle \nsubseteq \mathfrak{p}$, then $\mathfrak{p}$ appears zero times in any prime filtration of M.

Next, we deal with the case $\langle f \rangle \subseteq \mathfrak{p}$. Let 2.2 be a prime filtration of M as before. Observe that the annihilator of the R-module R/I for any ideal I is precisely I, so $\langle f \rangle = \operatorname{Ann} M \subseteq \operatorname{Ann} M_i \subseteq \operatorname{Ann}(M_i/M_{i-1}) = \mathfrak{p}_i$ for any i. Therefore, if $\mathfrak{q} \in \operatorname{Spec} R$ satisfies $\mathfrak{q} \supseteq \mathfrak{p}_i$ for any i, then it follows that $\mathfrak{q} \supseteq \langle f \rangle$. Conversely, if $\mathfrak{q} \nsupseteq \mathfrak{p}_i$ for any i, then it follows that for each i, there exists $s_i \in \mathfrak{p}_i = \operatorname{Ann}(M_i/M_{i-1})$ with $s_i \notin \mathfrak{q}$. In particular, $s_r m \in M_{r-1}$ for every $m \in M$. But then, $s_{r-1}(s_r m) \in M_{r-2}$ for every $m \in M$. Continuing inductively, we see that $\left(\prod_{i=1}^r s_i\right) m = 0$ for every m, so $\prod_{i=1}^r s_i \in \operatorname{Ann} M = \langle f \rangle$. Hence, $\prod_{i=1}^r s_i$ is an element of $\langle f \rangle$ that is not an element of $\mathfrak{q}$ since $\mathfrak{q}$ is a prime ideal and each s_i is not in $\mathfrak{q}$. In other words, $\mathfrak{q} \nsupseteq \langle f \rangle$. So we have shown that a prime ideal $\mathfrak{q}$ contains $\langle f \rangle$ if and only if $\mathfrak{q}$ contains $\mathfrak{p}_i$ for some i.

Since f is nonzero, R is an integral domain (so that the zero ideal is a prime ideal), and $\mathfrak{p}$ is height one, it follows that $\mathfrak{p} \supseteq \langle f \rangle$ is a minimal prime ideal over $\langle f \rangle$. By the previous paragraph, $\mathfrak{p} \supseteq \mathfrak{p}_{i_0}$ for some $1 \leq i_0 \leq r$. If $\mathfrak{p} \neq \mathfrak{p}_{i_0}$ then by the previous paragraph, $\mathfrak{p} \supsetneq \mathfrak{p}_{i_0} \supseteq \langle f \rangle$, contradicting the minimality of $\mathfrak{p}$ over $\langle f \rangle$. Therefore, $\mathfrak{p} = \mathfrak{p}_{i_0}$. If $\mathfrak{p}_{i_0}$ is not a minimal prime ideal among the prime ideals in the list $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$, then there is some other member of the list $\mathfrak{p}_{i_1}$ with $\mathfrak{p}_{i_1} \subsetneq \mathfrak{p}_{i_0}$, and hence by the previous paragraph, $\mathfrak{p} = \mathfrak{p}_{i_0} \supsetneq \mathfrak{p}_{i_1} \supseteq \langle f \rangle$, again contradicting the minimality of $\mathfrak{p}$ over $\langle f \rangle$. Therefore, $\mathfrak{p}$ is a minimal element from the list $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$.

Localizing our filtration at $\mathfrak{p}$, we obtain 2.3. As before, if $\mathfrak{p} = \mathfrak{p}_i$, then the quotient $(M_i)_\mathfrak{p}/(M_{i-1})_\mathfrak{p}$ is isomorphic to the fraction field of $R/\mathfrak{p}$ if $\mathfrak{p}_i \neq \mathfrak{p}$, then either $\mathfrak{p}_i = \langle 0 \rangle$ or $\mathfrak{p}_i \nsubseteq \mathfrak{p}$. But we have established that $\mathfrak{p}$ is a minimal element from the list $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$, so in fact if $\mathfrak{p}_i \neq \mathfrak{p}$, we must have $\mathfrak{p}_i \nsubseteq \mathfrak{p}$ and $\mathfrak{p}_i \neq \langle 0 \rangle$. In this case, as before, we obtain $(M_i)_\mathfrak{p}/(M_{i-1})_\mathfrak{p} \cong 0$. Therefore, 2.5 holds and just as before, the chain 2.3 becomes a composition series after possibly removing repeating submodules, establishing that the number of times $\mathfrak{p}$ appears in 2.2 depends only on M and $\mathfrak{p}$.

We worked hard to show the invariance of the number of times a height 1 prime appears in a prime filtration of $R/\langle f \rangle$. The reason we care about this is geometric. The height 1 prime ideals correspond exactly to codimension 1 subvarieties. It is to these subvarieties that we must assign multiplicities. For example, when we discussed multiplicities in Section 2.1, we discussed multiplicities of points on a curve. These points correspond to codimension 1 subvarieties of the curve. We will generalize the notion of multiplicity to codimension 1 subvarieties of schemes of arbitrary dimension.

Let $(X, \mathcal{O}_X)$ be a scheme and let $V \subseteq X$ be a subvariety of dimension 1. By the definition of a scheme, we can interpret V as a point $v \in X$. Suppose U is an affine neighborhood of v isomorphic to $\operatorname{Spec} R$. The stalk $\mathcal{O}_{X,v}$ of the structure sheaf at v, is isomorphic to the localization $R_\mathfrak{p}$ where $\mathfrak{p}$ is the prime ideal of R corresponding to the subvariety $U \cap V$ of U. Pick $f \in \mathcal{O}_{X,v}$. Abusing notation, we may consider f to be an element of $R_\mathfrak{p}$. Then, we may write $f = g/h$ where $g \in R$ and $h \in R \setminus \mathfrak{p}$. Note that since h is a unit in $R_\mathfrak{p}$, the ideal generated by f in $R_\mathfrak{p}$ is exactly the image of the ideal generated by g in R after localization at $\mathfrak{p}$. Therefore, since localization commutes with quotients, $R_\mathfrak{p}/\langle f \rangle \cong (R/\langle g \rangle)_\mathfrak{p}$. Hence it is well-defined to define the *multiplicity* of the subvariety V with respect to the function f to be the number of times $\mathfrak{p}$ appears in any prime filtration of the R-module $R/\langle g \rangle$.

In some sense, our work with prime filtrations is unnecessary. In the notation we used above, we essentially showed that a prime filtration of $R/\langle f \rangle$ becomes a composition series once we localize at the height 1 prime ideal $\mathfrak{p}$. So instead, we could have equivalently defined the multiplicity to call upon the length of a localized module rather than the number of times a prime ideal appears in a prime filtration. We made additional effort because [7] takes the definition of multiplicity to use prime filtrations while simultaneously referring to notions of "length" and "composition series". Our work shows that this abuse of terminology is justified.

If X is a normal scheme, then the definition of multiplicity is even simpler. In this case, the stalk $\mathcal{O}_{X,V}$ is a discrete valuation ring and so we can define the multiplicity of f to be its valuation in that ring. Nevertheless, all of these definitions capture the same notion of the *order of vanishing* of the function f on the subvariety V. For example, if $\langle g\rangle \nsubseteq \mathfrak{p}$ then g does not vanish on all of $\mathfrak{p}$ and so the order of vanishing of f on the subvariety $U \cap V$ determined by $\mathfrak{p}$ ought to be zero. Indeed, we saw earlier that if $\mathfrak{p} \nsupseteq \langle g\rangle$, then $\mathfrak{p}$ appears zero times in any prime filtration of $R/\langle g\rangle$. For this reason, the quantity is denoted $\operatorname{ord}_V f$.

Given this machinery, we are finally in a position to make definitions that parallel those of Section 2.1.

Definition 9. *Let X be a scheme. For every nonnegative integer k, a* ***cycle*** *of dimension k on X is is a formal finite linear combination $\sum_{i=1}^{m} a_i[V_i]$ where each V_i is a k-dimensional closed subvariety of X.*

As with divisors on curves, cycles of a fixed dimension form a free abelian group, denoted $Z_k(X)$. These cycles are analogous to cycles in singular homology. We will quotient these groups to form the Chow groups.

Note that if X is a variety and $V \subseteq X$ is a subvariety, then the fraction field of the local ring $\mathcal{O}_{X,V}$ is isomorphic to the localization of $\mathcal{O}_{X,V}$ at its zero ideal, which is the same as the stalk at the generic point of the scheme. This in turn is isomorphic to the field of rational functions on X. The upshot of this is that a rational function φ on X can be expressed as f/g for some $f, g \in \mathcal{O}_{X,V}$.

Definition 10. *Let X be a variety and $V \subseteq X$ a subvariety of codimension 1. Suppose φ is a nonzero rational function on X. Then, writing $\varphi = f/g$ for $f, g \in \mathcal{O}_{X,V}$, we define the* ***order of*** *φ* ***at*** *V to be $\operatorname{ord}_V \varphi := \operatorname{ord}_V f - \operatorname{ord}_V g$.*

Let $W \subseteq X$ be a subvariety of dimension $k+1$. If φ is a rational function on W, we define the ***divisor of*** *φ to be the cycle $\operatorname{div}\varphi := \sum_V (\operatorname{ord}_V \varphi)[V] \in Z_k(X)$ where the sum runs through all codimension 1 subvarieties $V \subseteq W$.*

Proof. The well-definedness of the definition of order follows analogously to that of Definition 7, via the same short exact sequence. We must additionally check that the sum $\sum_V (\operatorname{ord}_V \varphi)[V]$ is a finite sum. Indeed, since the subvariety W is quasicompact, there exists a finite affine open cover $\{U_i\}_{i=1}^{n}$ of W. Restricting φ to each open set, we obtain the rational functions $\varphi|_{U_i} \in \mathcal{O}_{U_i}(U_i)$. The zero loci of of these restrictions are closed in the U_i and the U_i are Noetherian topological spaces (so they can be decomposed into finitely many irreducible components), so the zero loci of each restriction $\varphi|_{U_i}$ only vanishes on finitely many components of U_i. Since there are finitely many sets U_i, this means that φ vanishes on only finitely many codimension 1 subvarieties V of W. □

The subgroup of $Z_k(X)$ generated by divisors of rational functions on X is denoted by $B_k(X)$. Just as in Section 2.1, it is interesting to study the quotient of $Z_k(X)$ by $B_k(X)$.

Definition 11. *Let k be a nonnegative integer and let X be a variety. The k^{th}* ***Chow group*** *is the quotient group $A_k(X) := Z_k(X)/B_k(X)$. Two cycles that determine the same equivalence class in a Chow group are said to be* ***rationally equivalent****.*

If X is a smooth projective curve, then unravelling the definitions will show that $Z_0(X) = \operatorname{Div} X$. Moreover, $\dim X = 1$ so the only 1-dimensional subvariety of X is itself. So Definition 10 implies that $A_0(X) = \operatorname{Pic} X$. So the Chow groups of a variety are a far-reaching generalization of the Picard group of a projective curve. However, the Chow groups are very difficult to compute in general. In fact, computing the Chow groups of even $\mathbb{A}^n$ and $\mathbb{P}^n$ is nontrivial. Despite this obstruction, Chow groups are useful for theoretical purposes.

As with anything resembling a homology theory, it is desirable to have notions of pullbacks and pushforwards of morphisms. It turns out that existence of pullbacks and pushforwards on Chow groups is a delicate question, and we will not provide all of the details of the full picture here (see [7]). However, we will state important cases in which pullbacks and pushforwards exist.

Let X be a scheme and $U \subseteq X$ an open subset. Let $i \colon U \hookrightarrow X$ be the inclusion. Then we can consider the map $Z_k(X) \to Z_k(U)$ given by $[Z] \mapsto [Z \cap U]$ for any k-dimensional subvariety $Z \subseteq X$. This map passes to rational equivalence because it is clear that $\operatorname{div} \varphi = \operatorname{div} \varphi|_U$ under this map. The map induced on the level of Chow groups is the pullback of the inclusion map, $i^* \colon A_*(X) \to A_*(U)$.

Another important example of a map that has a pullback on the Chow groups is a vector bundle. If X is a scheme and $\pi \colon E \to X$ is a vector bundle of rank r on X, then there are natural homomorphisms $A_k(X) \to A_{k+r}(E)$ given on cycles by $[V] \mapsto [\pi^{-1}(V)]$. Note that this map indeed passes to rational equivalence since this map sends $\operatorname{div} \varphi$ to $\operatorname{div} \pi^* \varphi$ for any rational function φ on a $(k+1)$-dimensional subvariety of X. So together, these maps determine a pullback on Chow groups, $\pi^* \colon A_*(X) \to A_{*+r}(E)$. In fact, it is true but difficult to show that this pullback is an isomorphism on the Chow groups. This gives another analogy with homology. A vector bundle on a topological space clearly deformation retracts to the space and so is homotopy-equivalent to that space. By the homotopy-invariance of homology, it follows that the homology of a vector bundle is isomorphic to that of the space it sits on top of.

Finally, we have pushforwards. The simplest case is the following: suppose that Y is a closed subscheme of X and $i \colon Y \hookrightarrow X$ is the inclusion. Then it is clear that the map $[Z] \mapsto [Z]$ for any subvariety $Z \subseteq Y$ passes to rational equivalence and thus induces a pushforward map on Chow groups $i_* \colon A_*(Y) \to A_*(X)$.

More generally, things get complicated. If X and Y are topological spaces, recall that we say that a function $X \to Y$ is *proper* if the preimage of every compact set is compact. If X is Hausdorff space and Y is a locally-compact Hausdorff (LCH) space, then the properness of a continuous map $f \colon X \to Y$ is equivalent to the following condition: for any continuous map $g \colon Z \to Y$, the canonical projection from the fiber product $X \times_Y Z$ to Z is a closed map. This motivates the scheme-theoretic definition of a proper morphism. If X and Y are separated schemes of finite type over a field, we say that a morphism $f \colon X \to Y$ is *proper* if for every scheme Z over Y, the canonical projection from the fiber product $X \times_Y Z$ to Z is a closed map.

A proper map is indeed a closed map as well. So if $f \colon X \to Y$ is a proper morphism of schemes, then for any subvariety Z, the image $f(Z)$ is a closed subvariety of Y with $\dim f(Z) \leq \dim Z$. Hence, we can define a pushforward map f_* at least on the level of cycles by setting $f_*([Z]) \coloneqq n_Z[f(Z)]$. The difficulty is in determining what the coefficients n_Z ought to be. It turns out that when $\varphi\colon Z \to W$ is a dominant morphism of varieties of the same dimension, one may interpret $K(Z)$ (the space of rational functions on Z) as a field extension of $K(W)$. The degrees of these field extensions (which are in fact the cardinalities of a general fiber of φ) will determine the coefficients n_Z. So we may define

$$n_Z \coloneqq \begin{cases} [K(Z) : K(f(Z))] & \dim f(Z) = \dim Z \\ 0 & \dim f(Z) < \dim Z. \end{cases} \tag{2.6}$$

One may show that indeed, this definition of f_* passes to rational equivalence and thus provides a map on the level of Chow groups.

2.3 Cartier Divisors and the Intersection Product

Now that we have completely defined the Chow rings as a group, we will work towards describing their ring structure. In fact, we will be unable to completely describe the multiplication operation in Chow rings. We will focus our attention to the special case of multiplication by elements of $A_{\dim X-1}(X)$, the codimension 1 Chow group. To accomplish this, we must first describe an important correspondence between line bundles on X and the elements of the codimension 1 Chow group.

Definition 12. *Let X be a scheme with $\dim X = n$. An element of $Z_{n-1}(X)$ is called a* ***Weil divisor on*** X*. Two Weil divisors are said to be* ***linearly equivalent*** *if they have the same equivalence class in $A_{n-1}(X)$. If D is a Weil divisor on X with nonnegative coefficients, then we say that D is* ***effective****, which we denote by $D \geq 0$. The* ***degree*** *of a Weil divisor is the sum of its coefficients. The* ***support*** *of a Weil divisor is the collection of codimension 1 subvarieties which appear with nonzero coefficients in the Weil divisor.*

Note that in Definition 10, we defined the divisor of a rational functions. Weil divisors that can be expressed as divisors of rational functions are called *principal*.

To each Weil divisor, we can associate a sheaf $\mathcal{O}_X(D)$. For every open subset $U \subseteq X$, this sheaf returns

$$\mathcal{O}_X(D)(U) \coloneqq \{\varphi \in K(X)^\times \colon \operatorname{div} \varphi|_U + D|_U \geq 0\} \cup \{0\}. \tag{2.7}$$

This sheaf records all the global rational functions whose poles and zeros are "controlled" by the divisor D in the sense that their orders and multiplicities are no worse than what D prescribes. This construction also passes to rational equivalence. That is, the Weil divisors D and E are linearly equivalent if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$. At least one direction of this statement is easy to see. Suppose D and E are linearly equivalent. Then by definition, if $D \neq E$ (in which case the conclusion is trivial), we have $D - E = \operatorname{div} \varphi$ for some $\varphi \in K(X)^\times$. Now for each open subset $U \subseteq X$, we can define the ring homomorphism $\Phi_U : \mathcal{O}_X(D)(U) \to \mathcal{O}_X(E)(U)$ defined by $\Phi_U(\psi) := \varphi\psi$. Indeed, Φ_U is a map into $\mathcal{O}_X(E)(U)$ because if $\operatorname{div}\psi|_U + D|_U \geq 0$, then we have

$$\operatorname{div}(\varphi\psi)|_U + E|_U = \operatorname{div}\psi|_U + \operatorname{div}\varphi|_U + E|_U = \operatorname{div}\psi|_U + D|_U \geq 0. \tag{2.8}$$

It is clear that each Φ_U is a ring isomorphism (since it has an inverse which is multiplication by $1/\varphi$). It is clear that these homomorphisms are compatible with restrictions, so the collection of Φ_U over all open subsets $U \subseteq X$ forms a sheaf isomorphism $\Phi : \mathcal{O}_X(D) \to \mathcal{O}_X(E)$. With a little more work, one may show that the converse is true as well.

We also note that the sheaf associated to the zero divisor $\mathcal{O}_X(0)$ is isomorphic to the structure sheaf $\mathcal{O}_X$. This boils down to showing that for each open subset $U \subseteq X$, there is an isomorphism $\mathcal{O}_X(0)(U) \to \mathcal{O}_X(U)$ that is compatible with restrictions. Indeed, if $U \subseteq X$ is open and $Y \subseteq U$ is a codimension 1 integral subscheme, and $\varphi \in \mathcal{O}_X(0)(U)$, then by definition $\operatorname{ord}_Y \varphi|_U \geq 0$. This means that φ is defined almost everywhere on Y so if V is the maximal domain for $\varphi|_U$, we will have $V \cap Y \neq \varnothing$ or $Y \nsubseteq U \setminus V$. Since Y is a closed subscheme of U that is not contained the closed subset $U \setminus V$, it follows that $U \setminus V$ has codimension at least 2. By the algebraic Hartog's lemma, it follows that φ extends to a regular function on all of U. Of course, conversely, every regular function U has an effective divisor or is zero, and thus belongs to $\mathcal{O}_X(0)(U)$. So there is an isomorphism as claimed.

We wish to state a correspondence between line bundles on X and some subcollection of the Weil divisors. To do this we must ask: for which Weil divisors D is the corresponding sheaf D locally free? If the sheaf is locally free, then Theorem 2 tells us that the sheaf represents some vector bundle on X. In some sense, the *Cartier divisors* are the answer to this question.

Definition 13. *Let X be a scheme of pure dimension. A* ***Cartier divisor on*** *X is a Weil divisor D on X such that there exists an open cover of X, say $\{U_\alpha\}_{\alpha \in A}$, such that for each $\alpha \in A$, we have $D|_{U_\alpha} = \operatorname{div}\varphi_\alpha|_{U_\alpha}$ for some $\varphi_\alpha \in K(X)^\times$.*

The group of Cartier divisors on a scheme X is denoted the $\operatorname{Pic} X$, called the *Picard group*. Indeed, if X is a smooth projective curve, this notion agrees with Definition 8. In fact, if X is any smooth scheme of pure dimension, then $\operatorname{Pic} X \cong A_{n-1}(X)$.

Succinctly, one may say that a Cartier divisor is simply a Weil divisor that is *locally principal.* It is important to note that in some sense, the open sets U_α and rational functions φ_α completely determine a Cartier divisor. Indeed, if $\alpha, \beta \in A$, then we should have $\operatorname{div}\varphi_\alpha|_{U_\alpha \cap U_\beta} = \operatorname{div}\varphi_\beta|_{U_\alpha \cap U_\beta} = D|_{U_\alpha \cap U_\beta}$. That is, $\operatorname{div}(\varphi_\alpha/\varphi_\beta)|_{U_\alpha \cap U_\beta} = 0$. In particular, $(\varphi_\alpha/\varphi_\beta)|_{U_\alpha \cap U_\beta} \in \mathcal{O}_X(0)(U_\alpha \cap U_\beta)$. But we have already shown that $\mathcal{O}_X(0) \cong \mathcal{O}_X$, so $(\varphi_\alpha/\varphi_\beta)|_{U_\alpha \cap U_\beta}$ is a regular function on $U_\alpha \cap U_\beta$. Similarly, $(\varphi_\beta/\varphi_\alpha)|_{U_\alpha \cap U_\beta}$ is a regular function on $U_\alpha \cap U_\beta$. This means that $(\varphi_\alpha/\varphi_\beta)|_{U_\alpha \cap U_\beta}$ is an invertible regular function on $U_\alpha \cap U_\beta$, which we denote by $(\varphi_\alpha/\varphi_\beta)|_{U_\alpha \cap U_\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)^\times$. So one may completely describe a Cartier divisor D as the data

$$D \cong \left\{ (U_\alpha, \varphi_\alpha)_{\alpha \in A} : \varphi_\alpha \in K(X)^\times, \frac{\varphi_\alpha}{\varphi_\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)^\times \text{ for all } \alpha, \beta \in A \right\} \Big/ \sim, \tag{2.9}$$

where the equivalence relation $\sim$ is given by $\{(U_i, \varphi_i)_{i\in I}\} \sim \{(V_j, \psi_j)_{j\in J}\}$ if and only if $\varphi_i/\psi_j \in \mathcal{O}_X(U_i \cap V_j)^\times$ for all $i \in I$ and $j \in J$.

As it turns out, the Cartier divisors are in correspondence with line bundles.

Theorem 3. *Let X be an integral scheme. There is a natural bijective correspondence between the Cartier divisor classes on X and isomorphism classes of line bundles on X.*

Proof. Suppose D is a Cartier divisor on X and let $\{U_\alpha\}_{\alpha\in A}$ be the open cover of X over which D is is principal. For each $\alpha \in A$, suppose that $D|_{U_\alpha} = \operatorname{div} \varphi_\alpha|_{U_\alpha}$. Then we have

$$\begin{aligned} \mathcal{O}_X(D)(U_\alpha) &= \{\psi \in K(X)^\times :\ \operatorname{div} \psi|_{U_\alpha} + \operatorname{div} \varphi_\alpha|_{U_\alpha} \geq 0\} \cup \{0\} \\ &= \{\psi \in K(X)^\times :\ \operatorname{div} (\psi\varphi_\alpha)|_{U_\alpha} \geq 0\} \cup \{0\} \\ &= \{\psi \in K(X) \colon \psi\varphi_\alpha \text{ is regular on } U_\alpha\}. \end{aligned} \tag{2.10}$$

This makes it clear that there is a map $\mathcal{O}_X(D)(U_\alpha) \to \mathcal{O}_X(U_\alpha)$ given by $\psi \mapsto \psi\varphi_\alpha$ and it is clear that this is an isomorphism. Hence, $\mathcal{O}_X(D)$ is a locally free sheaf of rank 1, which corresponds to a line bundle on X by Theorem 2. It is clear that this association descends to linear equivalence and thus a Cartier divisor class on X determines an isomorphism class of line bundles on X.

Conversely, let $\mathcal{L}$ be a line bundle on X. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X over which $\mathcal{L}$ is locally trivial. For each $\alpha \in A$, let $\varphi_\alpha \colon \mathcal{L}|_{U_\alpha} \to \mathcal{O}_X(U_\alpha)$ be the corresponding local trivialization. We may then consider the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1} \colon \mathcal{O}_X(U_\alpha \cap U_\beta) \to \mathcal{O}_X(U_\alpha \cap U_\beta)$. These maps are automorphisms of the ring $\mathcal{O}_X(U_\alpha \cap U_\beta)$, and are thus represented by multiplication by units in the ring, say $\psi_{\alpha,\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)^\times$. By construction, for $\alpha, \beta, \gamma \in A$, these elements obey the "cocycle condition" $\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Note that we may also interpret the regular functions $\psi_{\alpha,\beta}$ as rational functions in $K(X)$—since X is integral it is irreducible and thus the open set $U_\alpha \cap U_\beta$ is dense in X provided that it is nonempty. Since $U_\alpha \cap U_\beta \cap U_\gamma$ is dense in X, the cocycle condition which holds in that open set also holds on $K(X)$.

Fix $\alpha_0 \in A$ and consider the data $\{(U_\alpha, \psi_{\alpha,\alpha_0})_{\alpha\in A}\}$. Due to the cocycle condition, note that for every $\alpha, \beta \in A$ we have $\psi_{\alpha,\alpha_0}/\psi_{\beta,\alpha_0} = \psi_{\alpha,\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)^\times$. Therefore, modulo the equivalence relation $\sim$ we described above, equation 2.9 shows us that the data $\{(U_\alpha, \psi_{\alpha,\alpha_0})_{\alpha\in A}\}$ describes a Cartier divisor D on X. One can now see that $\mathcal{L} \cong \mathcal{O}_X(D)$. It can be shown that this is the inverse construction of our association of isomorphism classes of line bundles to each Cartier divisor class. □

Theorem 3 establishes that on on integral schemes, the Cartier divisors are precisely the Weil divisors D for which the associated sheaf $\mathcal{O}_X(D)$ is a line bundle. When we work modulo linear equivalence, we will often interpret a Cartier divisor as its corresponding line bundle. This interpretation of Cartier divisors is crucial for us to be able to define the special case of multiplication in the Chow ring.

We now come to defining the *intersection product*. Unfortunately, we cannot describe this in full generality. A complete description can be found in [6].

Definition 14. *Let X be a scheme, D a Cartier divisor on X, and $V \subseteq X$ a k-dimensional subvariety. Suppose $i\colon V \hookrightarrow X$ is the inclusion morphism. We define the* ***intersection product*** *to be $D \cdot V := i_*([i^*(\mathcal{O}_X(D))])$.*

Here, i^* is the pullback of the line bundle $\mathcal{O}_X(D)$ which exists because i is a closed immersion. Since V is a subvariety, Theorem 3 implies that isomorphism classes of line bundles on V correspond to Cartier divisor classes, and so $[i^*(\mathcal{O}_X(D))]$ is the divisor class that corresponds to the line bundle $i^*(\mathcal{O}_X(D)$. This is an element of $A_{k-1}(V)$, since it is the class of a Cartier divisor on V. To this divisor class, we apply the pushforward i_* which exists because closed immersions are proper. Hence, $D \cdot V \in A_{k-1}(X)$.

If V and W are subvarieties with dimension k and $n-1$, respectively, of a smooth scheme X with dimension n, and $V \nsubseteq W$ so that $\dim(V \cap W) = k-1$, then it turns out that the support of $W \cdot V$ is simply $[V \cap W]$. More precisely, the intersection product will capture the intersection $V \cap W$ along with some scheme-theoretic multiplicities. This is the motivation for why we call the operation of Definition 14 the "intersection product".

Using this intersection product, it turns out that we can partially define a multiplication operation on $A_*(X)$. First, we define the multiplication on the level of cycles. Since the k-dimensional subvarieties of X generate $Z_k(X)$, the intersection product naturally gives us a multiplication map $\operatorname{Pic} X \times Z_*(X) \to A_{*-1}(X)$. Note that moreover, Definition 14 depends not on D but on the isomorphism class of $\mathcal{O}_X(D)$. So by Theorem 3, our multiplication map descends to a bilinear map $\overline{\operatorname{Pic} X} \times Z_*(X) \to A_{*-1}(X)$ where $\overline{\operatorname{Pic} X}$ denotes rational equivalence classes of Cartier divisors. In fact, this product does give a multiplication map on the level of Chow groups.

Proposition 2. *Let X be an integral scheme. The intersection product descends to rational equivalence to induce a map $\overline{\operatorname{Pic} X} \times A_*(X) \to A_{*-1}(X)$.*

Proof. The proof hinges on the technical result that the intersection product is commutative. That is, given Cartier divisors D and E on X, we have $D \cdot E = E \cdot D$. This is a fact which we cannot prove here. One may refer to [6]. Let us assume this and suppose $W \subseteq X$ is a subvariety with $\dim W = k+1$ and $\varphi \in K(W)$. Then $\operatorname{div} \varphi$ is clearly a Cartier divisor on W (and also X if we interpret φ as a rational function on X) and so for any Cartier divisor D on X, we have $D \cdot \operatorname{div} \varphi = \operatorname{div} \varphi \cdot D = 0 \cdot D = 0$, where we invoke the fact that $\operatorname{div} \varphi$ is rationally equivalent to zero. Hence, we have shown that for any Cartier divisor D, the kernel of the intersection product with D, which is a map $Z_k(X) \to A_{k-1}(X)$, contains the kernel of the quotient map $Z_k(X) \to A_k(X)$. □

We have thus described a way to multiply elements of the Chow groups with certain types of elements of the Chow groups. This partially provides a ring structure to the Chow groups. For our purposes, this special case of multiplication will be sufficient.

Chapter 3

Chern Classes

With Theorem 1, we saw that determining when a vector bundle is trivial is a hard problem in general. In this chapter, we will develop objects called *Chern classes*, which attempt to detect when a vector bundle fails to be trivial. They do this by encoding information about where global sections of a vector bundle of fail to be independent. By the triviality criterion of Proposition 1, if certain Chern classes fail to vanish, this will imply that the vector bundle in question is nontrivial.

While we will state this geometric interpretation of Chern classes, we will not prove it in its full generality. A special case of this interpretation will show us that the *top* Chern class encodes information about the zero loci of global sections of a vector bundle. This interpretation of the top Chern class will ultimately prove to be useful for answering enumerative questions such as the question of the number of lines on a smooth cubic surface.

Throughout this chapter, we will routinely identify line bundles with Cartier divisors using Theorem 3.

3.1 Segre and Chern Classes

Before we define Chern classes, we must define *Segre classes*. These Segre classes can be expressed using the intersection product of Definition 14. We will then define Chern classes to be a sort of formal inverse to the Segre classes.

Definition 15. *Let X be an integral scheme and let $p\colon E \to X$ be a vector bundle of rank r on X. Let $\tilde{p}\colon \mathbb{P}(E) \to X$ be its projectivization. Suppose D_E is the Cartier divisor corresponding to the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. For every integer i, the i^{th}* ***Segre class of*** *E is a homomorphism $s_i(E)\colon A_*(X) \to A_{*-i}(X)$ given by*

$$s_i(E) \cdot \alpha \coloneqq \tilde{p}_*(D_E^{r+i-1} \cdot \tilde{p}^*(\alpha)). \tag{3.1}$$

The maps $\tilde{p}^*$ and $\tilde{p}_*$ are the pullback and pushforward maps described as in Section 2.2 and D_E^{r+i-1} is the intersection product of Definition 14 composed with itself $r+i-1$ times. There are several facts about Segre classes that we can state. For a full account of these facts, one may refer to [6] and [7]. We use the notation of Definition 15 as needed.

The first fact is that Segre classes are compatible with pullbacks. Let $f\colon X \to Y$ be a morphism such that a pullback $f^*\colon A_*(Y) \to A_*(X)$ exists. In Section 2.2, we noted that such a pullback exists if f is a vector bundle or if f is an open immersion. More generally, f has a pullback if it is a *flat morphism*, which is a notion we will not fully describe here. If E is a vector bundle on Y and $\alpha \in A_*(Y)$, then it turns out that

$$s_i(f^*(E)) \cdot f^*(\alpha) = f^*(s_i(E) \cdot \alpha). \tag{3.2}$$

The slogan here is that one may apply the Segre class and then pull back to X or equivalently, pull back to X and then apply the Segre class.

Segre classes also commute with each other. That is, if E_1 and E_2 are vector bundles on X, it is true that $s_i(E_1)s_j(E_2) = s_j(E_2)s_i(E_1)$.

When the vector bundle E is a line bundle, the Segre classes $s_i(E)$ are particularly easy to understand. In this case, the projectivization $\tilde{p}\colon \mathbb{P}(E) \to X$ is a rank 0 projective bundle on X. This means that the fibers are just singletons, so $\tilde{p}$ is an isomorphism (in fact, it is the identity). Then, as we saw in Section 1.3, the line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is just the tautological subbundle on $\mathbb{P}(E)$. In particular, when we canonically identify X with a subset of $\mathbb{P}(E)$ and restrict the line bundle of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ to X, we obtain a line bundle on X that is a subbundle of the line bundle E. In particular, this means that $\mathcal{O}_{\mathbb{P}(E)}(-1) = E$. Let D_E be the Cartier divisor corresponding to E. One can then show that $\mathcal{O}_{\mathbb{P}(E)}(1)$, the dual of the tautological subbundle, corresponds to the Cartier divisor $-E$. Since $\tilde{p}$ is the identity, Equation 3.1 reduces to $s_i(E) \cdot \alpha = (-1)^i D_E^i \cdot \alpha$. So up to a sign, the Segre classes of a line bundle are just the i-fold intersection product with the corresponding Cartier divisor.

Suppose $i < 0$ and V is a k-dimensional subvariety of X. Then $s_i(E)\cdot[V]$ is a $(k-i)$-dimensional cycle with nonzero coefficients corresponding only to subvarieties of V itself (due to the pullback and pushforward in the definition of the Segre class). But V has no subvarieties of dimension $k - i > k$ since $\dim V = k$. It follows that $s_i(E) \cdot [V] = 0$ so $s_i(E) = 0$. Similarly, it is clear that $Z_k(V)$ is the free group generated by $[V]$ itself so $s_0(E) \cdot [V] = n[V]$ for some integer n. With a little work, one can show that $n = 1$ so that $s_0(E) = \mathbf{1}$. A consequence of this fact is that $\tilde{p}_*\colon A_*(\mathbb{P}(E)) \to A_*(X)$ is surjective and $\tilde{p}^*\colon A_*(X) \to A_*(\mathbb{P}(E))$ is injective. Indeed, for any $\alpha \in A_*(X)$ we have $\alpha = s_0(E) \cdot \alpha = \tilde{p}_*(D_E^{r+i-1} \cdot \tilde{p}^*(\alpha))$, so every α is in the image of $\tilde{p}_*$ and $\ker \tilde{p}_* \cong 0$.

Now we define the Chern classes. Similar to Segre classes, they are operators on the level of the Chow rings of X given a vector bundle on X.

Definition 16. *Let X be an integral scheme and let $p\colon E \to X$ be a vector bundle of rank r on X. For $i \geq 0$, we define the i^{th} **Chern class of** E to be the homomorphism $c_i(E)\colon A_*(X) \to A_{*-i}(X)$*

satisfying the identity

$$\left(\sum_{i\geq 0} s_i(E)\right)\left(\sum_{i\geq 0} c_i(E)\right) = \mathbf{1}. \tag{3.3}$$

The sum of the Segre classes in equation 3.2 is called the *total Segre class* denoted $s(E)$ and similarly the sum of Chern classes in equation 3.2 is called the *total Chern class* denoted $c(E)$.

Equation 3.2 gives us an explicit way of writing Chern classes in terms of Segre classes. We may formally expand the equation and equate the graded pieces of both sides of the equation. The right hand side is simply the identity operator $\mathbf{1}$, which is an operator $A_*(X) \to A_{*-0}(X)$, and thus lives in the zeroth graded piece. Expanding the product of the total Segre and Chern classes, we obtain

$$\begin{aligned} c_0(E) &= \mathbf{1} \\ c_1(E) &= -s_1(E) \\ c_2(E) &= -s_2(E) + s_1(E)^2, \end{aligned} \tag{3.4}$$

and so on. Since the Chern classes are polynomials in the Segre classes, the statements we discussed for Segre classes have analogs for Chern classes. For instance, due to the formal identity $(\mathbf{1} - D + D^2 - \dots)(\mathbf{1} + D) = \mathbf{1}$, we have that when E is a line bundle, $c_1(E)$ corresponds to the intersection product with D_E and $c_i(E) = 0$ for $i > 1$. In other words, the total Chern class of a line bundle E is $\mathbf{1} + D_E$ where D_E is the Cartier divisor corresponding to E. In the next section, we will derive a similar explicit formula for the total Chern class of an arbitrary finite-rank vector bundle.

3.2 The Splitting Principle

The total Chern class (and total Segre class) also satisfy an important technical property which we will not prove here: they are multiplicative on exact sequences. That is, if we have have a short exact sequence of vector bundles on an integral scheme,

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0 \tag{3.5}$$

then $c(E) = c(E')c(E'')$ and $s(E) = s(E')s(E'')$. This property of Chern classes actually leads to an effective method of computing the Chern classes of vector bundles that are constructed from other more primitive vector bundles using “standard” operations, such as direct sums, duals, and so on. To see how, we will need the following result.

Proposition 3. *Let X be an integral scheme and let $p\colon E \to X$ be a vector bundle of rank r on X. There exists a scheme Y and a morphism of schemes $f\colon Y \to X$ such that f admits an injective pullback on the level of Chow groups and the pullback vector bundle $f^*(E)$ admits a filtration of subbundles*

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{r-1} \subsetneq E_r = f^*(E), \tag{3.6}$$

such that each successive quotient E_i/E_{i-1} is a line bundle on Y.

Proof. We induct on the rank of E. If E is of rank 1, then it is a line bundle and the conclusions are trivial. Suppose (a) is true for vector bundles of rank $r-1$ and let E have rank r. Let $p^\vee \colon E^\vee \to X$ be the dual bundle on X. Projectivizing this bundle, we obtain $\widetilde{p^\vee} \colon \mathbb{P}(E^\vee) \to X$. We remark that $\mathbb{P}(E^\vee)$ an integral scheme (see [1]).

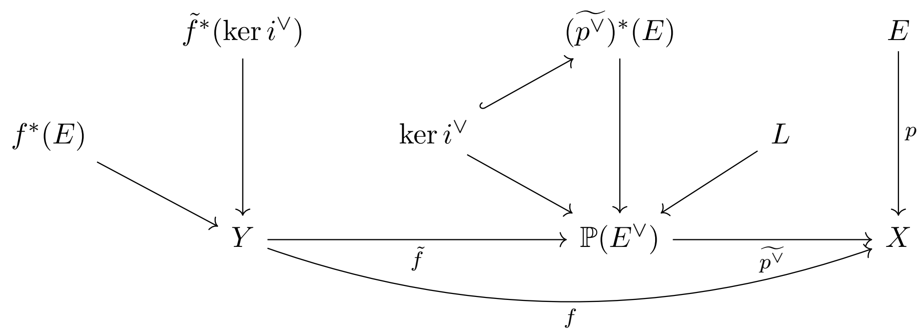
Now, one may consider the tautological subbundle $L^\vee \subseteq (\widetilde{p^\vee})^*(E^\vee)$. As a subbundle, there is an injective inclusion morphism of vector bundles $i \colon L^\vee \hookrightarrow (\widetilde{p^\vee})^*(E^\vee)$. Because $\mathbb{P}(E^\vee)$ is integral, it is reduced, so the cokernel of an injective morphism of vector bundles on $\mathbb{P}(E^\vee)$ is indeed a vector bundle on $\mathbb{P}(E^\vee)$. Therefore, we have a the following short exact sequence of vector bundles on $\mathbb{P}(E^\vee)$:

$$0 \longrightarrow L^\vee \xrightarrow{\ i\ } (\widetilde{p^\vee})^*(E^\vee) \longrightarrow \operatorname{coker} i \longrightarrow 0 \tag{3.7}$$

Dualizing this short exact sequence, we obtain the short exact sequence of vector bundles on $\mathbb{P}(E^\vee)$:

$$0 \longrightarrow \ker i^\vee \longrightarrow (\widetilde{p^\vee})^*(E) \xrightarrow{\ i^\vee\ } L \longrightarrow 0 \tag{3.8}$$

By the rank-nullity theorem (the additivity of dimension on exact sequences of vector spaces), we have that $\operatorname{rank} \ker i^\vee = \operatorname{rank} E - \operatorname{rank} L = \operatorname{rank} E - 1 = r - 1$. So $\ker i^\vee$ is a vector bundle of rank $r-1$ on the integral scheme $\mathbb{P}(E^\vee)$. By the inductive hypothesis, there exists a morphism $\tilde{f} \colon Y \to \mathbb{P}(E^\vee)$ such that $\tilde{f}^*(\ker i^\vee)$ is a line bundle on the scheme Y that admits a filtration by subbundles where successive quotients are line bundles on Y. Define $f \coloneqq \widetilde{p^\vee} \circ \tilde{f} \colon Y \to X$. The situation thus far is given by the following diagram.



(3.9)

We can extend the filtration of $\tilde{f}^*(\ker i^\vee)$ into a filtration of $f^*(E)$ to obtain

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{r-1} = \tilde{f}^*(\ker i^\vee) \subsetneq f^*(E). \tag{3.10}$$

In this filtration, it is clear that the successive quotients E_i/E_{i-1} are line bundles for $0 \le i \le r-1$. That $f^*(E)/E_{r-1}$ is a line bundle follows from the short exact sequence 3.8 which implies that $f^*(E)/E_{r-1} = \tilde{f}^*[(\widetilde{p^\vee})^*(E)]/\tilde{f}^*(\ker i^\vee) \cong \tilde{f}^*(L)$, where the pullback $\tilde{f}^*$ commutes with the quotient because pullback is an exact functor on locally free sheaves.

It is clear that f has a Chow-pullback (a pullback on the level of Chow groups). This is because it is the composition of $\tilde{f}$ which has a Chow-pullback by the inductive hypothesis, and $\widetilde{p^\vee}$, because this is a projective bundle map. By the inductive hypothesis, $\tilde{f}$ has an injective Chow-pullback. Moreover, we noted in Section 3.1 that a consequence of the fact that the zeroth Segre class is the identity is that the Chow-pullback of a projective bundle map is injective, so the Chow-pullback of $\widetilde{p^\vee}$ is injective as well. It follows that the Chow-pullback of f is injective. □

Proposition 3 gives us a powerful way to express the total Chern class of a vector bundle. Let $p\colon E \to X$ be a vector bundle of rank r on an integral scheme. Using the notation of Proposition 3, we can see that we have several exact sequences of the form

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow L_i \longrightarrow 0 \tag{3.11}$$

where L_i is the line bundle resulting from the i^{th} successive quotient. Since the total Chern class is multiplicative on exact sequences, this implies that

$$c(f^*(E)) = c(E_r) = c(E_{r-1})c(L_r) = c(E_{r-2})c(L_{r-1})c(L_r) = \cdots = \prod_{i=1}^{r} c(L_i). \tag{3.12}$$

Recall that in Section 3.1 we saw how to interpret the total Chern class of a line bundle. For each i, if D_i is the Cartier divisor corresponding to the line bundle L_i, we have that $c(L_i) = \mathbf{1} + D_i$. Hence, equation 3.12 becomes

$$c(f^*(E)) = \prod_{i=1}^{r} (\mathbf{1} + D_i). \tag{3.13}$$

The Cartier divisors D_i which can be interpreted as the Chern classes $c_1(L_i)$ are known as the *Chern roots* of the vector bundle E. Of course, this terminology is misleading, since the Chern roots certainly depend on the morphism $f\colon Y \to X$ we use. Proposition 3 does not imply that the morphism f is unique.

One may observe that equation 3.13 is only capable of expressing the total Chern class of the pullback bundle $f^*(E)$ and not E itself. In the special case that for each i we have $L_i = f^*(L'_i)$ for some line bundles L'_i on X. Hence, equation 3.12 becomes

$$c(f^*(E)) = \prod_{i=1}^{r} c(f^*(L'_i)) = f^*\left(\prod_{i=1}^{r} c(L'_i)\right). \tag{3.14}$$

where the second equality comes from the fact that Chern classes commute with pullbacks. Finally, by Proposition 3, we know that f^* is injective, so in fact equation 3.14 implies that

$$c(E) = \prod_{i=1}^{r} c(L'_i) = \prod_{i=1}^{r} \left(\mathbf{1} + D_{L'_i}\right), \tag{3.15}$$

However, in general it is not true that the line bundles L_i on Y are pullbacks of line bundles on X, so we cannot express $c(E)$ in a "Chern decomposition" like we were able to for $c(f^*(E))$. Nonetheless, equation 3.13 still makes makes more concrete what is otherwise a rather opaque abstract operator. The Chern decomposition of $f^*(E)$ along with the injectivity of f^* is capable of elucidating properties of the total Chern class of E. For example, we can see that $c_i(E) = 0$ for all $i > r = \operatorname{rank} E$. Note that by expanding the product in 3.13, we can see that $c(f^*(E))$ has no graded pieces in degree higher than r, so $c_i(f^*(E)) = 0$ for $i > r$. This implies that for every $\alpha \in A_*(X)$, we have $c_i(f^*(E)) \cdot f^*(\alpha) = 0$. But as we saw in Section 3.1, Segre classes (and thus Chern classes) commute with pullbacks, so this becomes $f^*(c_i(E) \cdot \alpha) = 0$ for every $\alpha \in A_*(X)$. Since f^* is injective, it follows that $c_i(E) = 0$.

In general, suppose we wish to show a polynomial relation amongst the Chern classes $c_i(E)$. Since the Chern classes vanish beyond the r^{th} Chern class, this amounts to showing that for some polynomial P in r variables, we have $P(c_1(f^*(E)), \ldots, c_r(f^*(E))) = 0$. Since Chern classes commute with pullbacks, and the pullback is a homomorphism, this implies that $f^*(P(c_1(E), \ldots, c_r(E)))$ $= 0$. The injectivity of f^* thus implies that $P(c_1(E), \ldots, c_r(E)) = 0$. Hence, polynomial relations amongst the Chern classes of the pullback $f^*(E)$ will imply the same relations amongst the Chern classes of E. This means, for the purposes of establishing polynomial relations amongst the Chern classes of E, we might as well assume that E itself admits a filtration of the form 3.6 so that the Chern roots can be interpreted as multiplication by genuine divisors on X.

The Chern decomposition thus allows us to study the Chern class of a vector bundle that is constructed from vector bundles whose Chern classes are known by relating the Chern roots of the new bundle with those of the old bundle. For instance, suppose E is a vector bundle on X with Chern roots $\{\alpha_i\}_i$. Then it is fairly straightforward to show that $E^\vee$ has Chern roots $\{-\alpha_i\}_i$. Similarly, the vector bundle $\operatorname{Sym}^k E$, which is the k^{th} symmetric power of E, has Chern roots $\{\alpha_{i_1} + \cdots + \alpha_{i_k}\}_{i_1 \leq \cdots \leq i_k}$. Similar formulas for Chern roots exist for exterior powers and tensor products.

3.3 Linear Independence of Global Sections

So far, we have been treating Chern classes as abstract operators. In Section 3.2, we discussed ways to think of and manipulate Chern classes formally. In this section, we provide some context towards why we actually care about Chern classes. Specifically we will state (but not prove) a geometric interpretation of Chern classes. We will see that geometrically, Chern classes are linked to the problem of determining how nontrivial a vector bundle is.

Let us return to the topological setting. The reference for what follows is [10]. Here, X is a topological space and $p\colon E \to X$ is a complex vector bundle. We insist that our vector bundles are complex, since if we deal with real vector bundles we run into issues of orientability and we will need to start using $\mathbb{Z}_2$-homology. The analog of Chern classes in the setting of real vector bundles are called *Stiefel-Whitney classes*.

In this setting, the Chern classes $c_i(E)$ are elements of the cohomology group $H^{2i}(X; \mathbb{Z})$. These cohomology classes satisfy the property that they commute with pullbacks and $c_i(E) = 0$ when $i > \operatorname{rank} E$. The rough idea of the geometric interpretation of the Chern classes is as follows. Recall that in Proposition 1, we noted that E is trivial if and only if there are $\operatorname{rank} E$ sections of E that are linearly independent on all of X. After a change of basis, this is equivalent to insisting that there are $\operatorname{rank} E$ sections of E that are orthonormal on all of X. So one may ask for each $1 \leq k \leq n$, whether there are k orthonormal sections of E.

Let us further assume that X is a CW complex. Let the i-skeleton of X be X_i. Suppose that we have k orthonormal sections on the subspace $X_{i-1} \subseteq X$. We may ask what obstructions can occur if we try to extend those k sections to X_i while preserving orthonormality. Each i-cell of X comes with an attaching map $\varphi_i \colon D^i \to X$ where D^i is the closed i-dimensional disk and $\varphi_i|_{\partial D^i}$ has image contained in X_{i-1}. Theorem 1 implies that all vector bundles over a contractible space are trivial. The disk D^i is clearly contractible, so the pullback $\varphi_i^*(E)$ on D^i is trivial. The triviality of this pullback ensures that the k orthonormal sections we have on X_{i-1} and get pulled back to k orthonormal sections on ∂D^k. Said another way, we have induced a map $\psi \colon \partial D^i \to V_k(\mathbb{C}^r)$, where $V_k(\mathbb{C}^r)$ is the Stiefel manifold described (in the real case) in Section 1.2.

Our desire of extending our k orthonormal sections to the i-skeleton is equivalent to continuously extending the map ψ to a map Ψ on the full disk D^i. It turns out that this is possible if and only if the map ψ is nullhomotopic (see [3]). The geometric intuition for this fact is the following. The disk D^i is made up of concentric shells homeomorphic to ∂D^i. If there is a homotopy $H(x, t)$ between ψ and a constant map, then we can extend ψ to the disk D^i by declaring $\Psi(x, t) \coloneqq H(x, 1 - t)$ where (x, t) corresponds to the point in the disk that is on the shell of radius t with an "angular coordinate" x. This declaration is well-defined at the origin because $H(x, t)$ is a homotopy with a constant map. In other words, as we progress through the nullhomotopy, we define the extension of ψ over smaller and smaller concentric shells in the disk, until finally we reach the center of the disk at the end of the nullhomotopy, where the extension of ψ is defined to be the image of the constant map that ψ was homotopic to.

Since ∂D^i is homeomorphic to S^{i-1}, it is clear that the map $\psi \colon \partial D^i \to V_k(\mathbb{C}^r)$ is nullhomotopic if (but maybe not only if) the homotopy group $\pi_{i-1}(V_k(\mathbb{C}^r))$ vanishes. For example, if $k = 1$, then we are asking when a single section on the $(i-1)$-skeleton can be extended to an i-cell, and since $V_1(\mathbb{C}^r) \cong S^{2r-1}$. We have just shown that such a section can certainly be extended when $\pi_{i-1}(S^{2r-1})$ is nontrivial. In fact, the first nonzero homotopy group of a sphere S^n is $\pi_n(S^n)$, so when $i < 2r$, there are no obstructions to extending our single section to the i-cell. However, for arbitrary k, we are forced to deal not with spheres but rather the more complicated Stiefel manifolds $V_k(\mathbb{C}^r)$. It turns out that the first nonvanishing homotopy group of the Stiefel manifold is $\pi_{2r-2k+1}(V_k(\mathbb{C}^r))$. So when $i < 2r - 2k + 2$, there are no obstructions to extending k sections on X_{i-1} to any of the i-cells in question (and thus all of X_i).

Suppose then $i = 2r - 2k + 2$. We can construct a map on the set of i-cells of X. Each i-cell is sent to the nontrivial element of $\pi_{2r-2k+1}(V_k(\mathbb{C}^r))$ that somehow represents an "obstruction" to the task of extending our k sections on X_{i-1} to that particular i-cell. The data of these maps can be interpreted as an i-dimensional cellular cochain, with coefficients in $\pi_{2r-2k+1}(V_k(\mathbb{C}^r))$. That is, this data determines an element of the cohomology $H^{2r-2k+2}(X;\pi_{2r-2k+1}(V_k(\mathbb{C}^r)))$. Moreover, one may show that $\pi_{2r-2k+1}(V_k(\mathbb{C}^r))$ is canonically isomorphic to $\mathbb{Z}$, so we have determined an element of the cohomology $H^{2r-2k+2}(X;\mathbb{Z})$. Up to a sign, this is precisely the Chern class $c_{r-k+1}(E)$!

On a CW complex, the Chern classes thus measure obstructions to extending sections on one skeleton to the next. There is a similar interpretation in the algebro-geometric setting. Suppose E is a vector bundle of rank r on a scheme X and $v_1,\ldots,v_{r+1-k}$ are global sections of E. Let $Z \subseteq X$ be the scheme-theoretic locus where the sections $v_1,\ldots,v_{r+1-k}$ are linearly dependent. If D has codimension k in X, then $[Z] = c_k(E)\cdot[X]$, where $[Z]$ corresponds to the Chow-class of the locus Z and $[X]$ is the Chow-class of X. In other words, $c_k(E)$ encodes the complexity of the subspace over which the vector bundle E does not have linearly independent "generic" sections. The more nontrivial the Chern classes are, the more complicated is the subspace over which generic sections fail to be linearly independent.

In fact, Segre classes have a similar interpretations. Up to a sign, we have $s_k(E)\cdot[X] = \pm[W]$, where W is the scheme-theoretic locus of points $x \in X$ such that for each $x \in X$, the $r+1-k$ generic sections fail to span the fiber of E over x. So Chern classes measure where sections fail to be linearly independent, Segre classes measure where sections fail to span the fiber, and together the Chern and Segre classes measure in some sense how much the sections fail to form a basis of each fiber.

A full proof of this geometric interpretation of Chern and Segre classes in the algebro-geometric setting is not possible here. One may refer to [6] for those details. However, we will be able to describe most of the proof of a special case of this geometric interpretation of Chern classes. In particular, we are able to say more about the *top Chern class*, that is the Chern class of degree $\operatorname{rank} E = r$. In this case, our geometric interpretation of the Chern class reduces to a statement about the linear dependence of a single generic global section of E. Since a single vector is linearly dependent if and only if it is the zero vector, the top Chern class thus really tells us about the zero locus of a single global section.

Theorem 4. *Let E be a vector bundle of rank r on an integral scheme X of dimension n. Let s be a global section of E, with scheme-theoretic zero locus Z. If this zero locus has dimension $n-r$, then $[Z] = c_r(E)\cdot[X]$.*

Proof. Let $f\colon Y \to X$ be the morphism given by Proposition 3. Note that $f^*(E)$ has section $f^*(s)$ which vanishes on $f^*(Z)$. Then if $i'\colon Z \hookrightarrow X$ and $j'\colon f^*(Z) \hookrightarrow Y$ are the inclusions, then $j'_*f^*([Z]) = c_r(f^*(E))\cdot[Y]$ if and only if $f^*i'_*([Z]) = f^*(c_r(E)\cdot[X])$, because Chern classes commute with pullbacks. Since f^* is injective, the conclusion will follow if either equality holds. Therefore, we may assume without loss of generality that E itself admits a good filtration in the sense of Proposition 3.

We induct on r. The base case is clear: when $r=1$, E is a line bundle and $c_1(E)$ corresponds to Cartier divisor class of the zero locus of a generic section by Theorem 3. Suppose the claim is true for vector bundles of rank up to $r-1$. Since we are assuming that E admits a good filtration, there exists a short exact sequence of vector bundles on X:

$$0 \longrightarrow F \longrightarrow E \xrightarrow{\ p\ } L \longrightarrow 0 \tag{3.16}$$

Suppose the section $p(s)$ of L has a zero locus W. Making restrictions to W, we have the short exact sequence of vector bundles on W:

$$0 \longrightarrow F|_W \longrightarrow E|_W \xrightarrow{\ p\ } L|_W \longrightarrow 0 \tag{3.17}$$

Because $p(s)|_W$ is the zero section of $L|_W \cong E|_W/F|_W$ by definition, it follows that the section $s|_W$ of $E|_W$ descends to a section of $F|_W$ which we will also denote by $s|_W$. Note that $s|_W$ vanishes on Z and $Z \subseteq W$ since $p(s)$ will vanish where s does.

Let $i\colon Z \hookrightarrow W$ and $j\colon W \hookrightarrow X$ be the inclusions. Since we have a chain of subspaces $Z \subseteq W \subseteq X$, we will use subscripts to keep track of which Chow group each divisor class belongs to. Recall that Z is the zero locus of $s|_W$ and $\dim Z = n-r$ by assumption. We also have that $\dim W = n-1$ since $p(s)$ is a "generic" nonzero section of the line bundle L. Hence, $\dim Z = \dim W - r + 1$. It follows that the vector bundle $F|_W$ satisfies conditions for the inductive hypothesis: it is a rank $r-1$ vector bundle on W with a global section whose zero locus satisfies the correct dimension condition. By the inductive hypothesis,

$$i_*([Z]_Z) = c_{r-1}(F|_W) \cdot [W]_W. \tag{3.18}$$

But one may note that $[W]_W$ is the j-pullback of the zero locus of $p(s)$, which by the inductive hypothesis applied to the rank 1 case, is precisely $j^*(c_1(L) \cdot [X])$. Moreover, $F|_W = j^*(F)$. Hence, 3.18 becomes

$$i_*([Z]_Z) = c_{r-1}(F|_W) \cdot [W]_W = c_{r-1}(j^*(F)) \cdot j^*(c_1(L) \cdot [X]) = j^*(c_{r-1}(F)c_1(L) \cdot [X]), \tag{3.19}$$

where the last equality comes from the fact that Chern classes commute with pullbacks. However, one may see that the divisor class of a line bundle is the class of the divisor of the zero locus of a generic section from Theorem 3. Therefore, $c_1(L)\cdot[X] = [W]_X$ and it follows that $c_{r-1}(F)c_1(L)\cdot[X]$ is supported in W. This means that when we apply the pushforward j_* to both sides of equation 3.19, the composition $j_* \circ j^*$ will behave as the identity in our case, and we obtain

$$j_*(i_*([Z]_Z)) = c_{r-1}(F)c_1(L) \cdot [X] = c_r(E) \cdot [X], \tag{3.20}$$

where we obtain the last equality from the multiplicativity of Chern classes on exact sequences applied to 3.16. This completes the induction. □

We have now developed, or at least stated, all of the tools necessary to count how many lines are on smooth cubic surfaces in $\mathbb{P}^3$. We will employ the interpretation of the top Chern class provided by Theorem by considering a vector bundle whose zero locus corresponds precisely to the collection of lines on a given smooth cubic surface. We work through the details in the next chapter.

Chapter 4

Counting Lines on a Cubic Surface

In this chapter, we apply the tools we have developed to solve the following enumerative problem.

How many lines are on a smooth cubic surface in $\mathbb{P}^3$?

Strictly speaking, the approach we will take in Section 4.1 will be sufficient to completely answer this question independently of our theory of Chern classes. However, by applying Chern classes in Section 4.2, we will gain a better understanding of where the precise numerical answer comes from and a more general strategy that can be employed to solve similar enumerative problems. Most of the arguments we will provide in this chapter can be found in [7].

4.1 The Dimension Criterion

We will answer our question by constructing a vector bundle on a space whose zero locus captures the collection of lines on some cubic surface. The number of lines on that cubic surface will thus be the degree of the divisor of the zero locus. By applying Theorem 4, we will compute this degree.

In order to apply Theorem 4, we must ensure that our zero locus has the correct dimension. In our case, it suffices to show that the number of lines on a generic smooth cubic in $\mathbb{P}^3$ is finite. We first show that the number of lines on a cubic surface is independent of the cubic chosen.

Proposition 4. *The number of lines on a smooth cubic surface in $\mathbb{P}^3$ is independent of the cubic chosen.*

Proof. A cubic surface in $\mathbb{P}^3$ is determined by the zero locus of a homogeneous cubic polynomial in four variables. The number homogeneous cubic monomials can be computed using simple combinatorics. It is $\binom{6}{3} = 20$. These monomials form a basis for the space of homogeneous cubic polynomials in four variables (if we include zero as a cubic). So that space can be identified with the affine space $\mathbb{A}^{20}$. But by homogeneity, the zero locus of any of the cubic polynomials is invariant under the operation of multiplying the cubic polynomial by a nonzero scalar. So the space of cubic surfaces in $\mathbb{P}^3$ can be identified with the projective space $\mathbb{P}^{19}$. This is the so-called *moduli space* of cubic surfaces in $\mathbb{P}^3$.

Now we may define the *incidence correspondence* $M := \{(L, X) \in G(1, \mathbb{P}^3) \times \mathbb{P}^{19} \colon L \subseteq X\}$, where $G(1, \mathbb{P}^3)$ is the Grassmannian of lines in $\mathbb{P}^3$ and we interpret the statement $L \subseteq X$ to mean that the line L is on the cubic surface represented by the point in the moduli space $\mathbb{P}^{19}$. Thus, this incidence correspondence consists of all pairs of a line and a cubic such that the line lies on the cubic. Let $\pi \colon M \to \mathbb{P}^{19}$ be the projection map. Observe that by construction, for any cubic surface in $\mathbb{P}^3$, if that cubic surface is represented by $X \in \mathbb{P}^{19}$, then the number of lines on that cubic is precisely the cardinality of the fiber $\pi^{-1}(\{X\})$.

We now describe a coordinate system in which we will perform calculations. Recall that the Grassmannian $G(1, \mathbb{P}^3)$ is a 4-dimensional projective variety (via the Plücker embedding). Suppose $\mathbb{P}^3$ is given coordinates $[x_0 : x_1 : x_2 : x_3]$. For any fixed $L_0 \in G(1, \mathbb{P}^3)$, one may performa change of coordinates so that L_0 is the locus given by the equations $x_2 = x_3 = 0$. Then, one may construct an affine neighborhood $\mathbb{A}^4 \subseteq G(1, \mathbb{P}^3)$ of L_0 given by sending the affine point $(a_2, a_3, b_2, b_3) \in \mathbb{A}^4$ to the point in $G(1, \mathbb{P}^3)$ corresponding to the line through the points $(1, 0, a_2, a_3)$ and $(0, 1, b_2, b_3)$. By our construction of the moduli space $\mathbb{P}^{19}$ of smooth cubics, we note that the projective coordinates in that space c_α are indexed by multi-indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \geq 0$ and $\sum_{i=0}^{3} \alpha_i = 3$. Let $\mathscr{A}$ be the collection of these multi-indices. These multi-indices α keep track of which monomial corresponds to the coefficient-coordinate c_α. Hence, locally on M we may describe points with the coordinates $(a, b, c) = (a_2, a_3, b_2, b_3, (c_\alpha)_{\alpha \in \mathscr{A}})$. For the coordinate c, we let $X_c \subseteq \mathbb{P}^3$ denote the actual geometric surface corresponding to the zero locus of the cubic corresponding to the point $c \in \mathbb{P}^{19}$.

The first result we must show is that M is a smooth variety. Pick a point in M and write it in local coordinates as (a, b, c). Since $(a, b, c) \in M$, we have that

$$s[1 : 0 : a_2 : a_3] + t[0 : 1 : b_2 : b_3] = [s : t : sa_2 + tb_2 : sa_3 + tb_3] \in X_c \tag{4.1}$$

for all $s, t \in \mathbb{C}$. Here, we are simply unravelling the definition of the incidence correspondence using our coordinates. Going further, 4.1 implies that

$$\sum_{\alpha \in \mathscr{A}} c_\alpha s^{\alpha_0} t^{\alpha_1} (sa_2 + tb_2)^{\alpha_2} (sa_3 + tb_3)^{\alpha_3} = 0, \tag{4.2}$$

for all $s, t \in \mathbb{C}$. Since $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 3$, we can expand and group the sum above by $s^i t^{3-i}$ terms to rewrite 4.3 as

$$\sum_{i=0}^{3} s^i t^{3-i} F_i(a, b, c) = 0 \tag{4.3}$$

for all $s, t \in \mathbb{C}$, where the F_i are some polynomials. By construction, these F_i are linear in the c_α. For each $i \in \{0, 1, 2, 3\}$, we claim that $c_{(i,3-i,0,0)}$ occurs as a term in the polynomial F_i. Indeed, $c_{(i,3-i,0,0)}$ occurs in F_i. Suppose it occurs in F_j. Then considering the origin of the term

$s^j t^{3-j} F_j(a,b,c)$, it follows that $j = i$. Note moreover that $c_{(i,3-i,0,0)}$ appears in F_i with coefficient 1. Therefore, the equation $F_i(a,b,c) = 0$ can be rearranged to $c_{(i,3-i,0,0)} = G_i(a,b,c)$ for some polynomial G_i. Observe also that the polynomial G_i only depends on c_α with $\alpha_2 > 0$ or $\alpha_3 > 0$, since $c_{(i,3-i,0,0)}$ occurs precisely in F_i, so G_j will not depend on it if $j \neq i$ and even G_i will not depend on it by construction.

Therefore, if we endow the variety $\mathbb{A}^4 \times \mathbb{P}^{15}$ with the coordinates $(a_2, a_3, b_2, b_3, (c_\alpha)_{\alpha \in \mathscr{B}})$, where $\mathscr{B} \subsetneq \mathscr{A}$ is the subcollection of multi-indices α with $\alpha_2 > 0$ and $\alpha_3 > 0$, we have found an isomorphism between $\mathbb{A}^4 \times \mathbb{P}^{15}$ and an open subvariety of M, given by the equations $c_{(i,3-i,0,0)} = G_i(a,b,c)$. Since $\mathbb{P}^{15}$ contains a copy of $\mathbb{A}^{15}$, we have shown that M admits an open cover by the affine spaces $\mathbb{A}^4 \times \mathbb{A}^{15} = \mathbb{A}^{19}$. Hence, M is a smooth 19-dimensional variety.

Next, let us suppose our point $(a,b,c) \in M$ is such that X_c is a smooth cubic surface. Changing coordinates, we may assume without loss of generality that $a = b = 0$. Let f_c denote the homogeneous cubic polynomial defining the surface X_c. By construction, we have

$$\sum_{i=0}^{3} s^i t^{3-i} F_i(a,b,c) = f_c(s, t, sa_2 + tb_2, sa_3 + tb_3). \tag{4.4}$$

Therefore, by the chain rule we have

$$\begin{aligned} \frac{\partial}{\partial a_2}\left(\sum_{i=0}^{3} s^i t^{3-i} F_i(a,b,c)\right)\Bigg|_{(0,0,c)} &= \frac{\partial}{\partial a_2} f_c(s, t, sa_2 + tb_2, sa_3 + tb_3)\Big|_{(0,0,c)} \\ &= s\frac{\partial f_c}{\partial x_2}(s,t,0,0). \end{aligned} \tag{4.5}$$

Let $F = [F_0, F_1, F_2, F_3]^T$ and consider the following submatrix of the Jacobian DF:

$$J := \begin{bmatrix} \frac{\partial F_0}{\partial a_2} & \frac{\partial F_0}{\partial a_3} & \frac{\partial F_0}{\partial b_2} & \frac{\partial F_0}{\partial b_3} \\ \frac{\partial F_1}{\partial a_2} & \frac{\partial F_1}{\partial a_3} & \frac{\partial F_1}{\partial b_2} & \frac{\partial F_1}{\partial b_3} \\ \frac{\partial F_2}{\partial a_2} & \frac{\partial F_2}{\partial a_3} & \frac{\partial F_2}{\partial b_2} & \frac{\partial F_2}{\partial b_3} \\ \frac{\partial F_3}{\partial a_2} & \frac{\partial F_3}{\partial a_3} & \frac{\partial F_3}{\partial b_2} & \frac{\partial F_3}{\partial b_3} \end{bmatrix}. \tag{4.6}$$

The calculation in equation 4.5 implies that the first column of the matrix $J(0,0,c)$ are the $s^i t^{3-i}$-coefficients of the polynomial $s(\partial f_c/\partial x_2)(s,t,0,0)$. Similarly, the other columns of $J(0,0,c)$ are coefficients of

$$\frac{\partial}{\partial a_3}\left(\sum_{i=0}^{3} s^i t^{3-i} F_i(a,b,c)\right)\bigg|_{(0,0,c)} = s\frac{\partial f_c}{\partial x_3}(s,t,0,0)$$
$$\frac{\partial}{\partial b_2}\left(\sum_{i=0}^{3} s^i t^{3-i} F_i(a,b,c)\right)\bigg|_{(0,0,c)} = t\frac{\partial f_c}{\partial x_2}(s,t,0,0) \tag{4.7}$$
$$\frac{\partial}{\partial b_3}\left(\sum_{i=0}^{3} s^i t^{3-i} F_i(a,b,c)\right)\bigg|_{(0,0,c)} = t\frac{\partial f_c}{\partial x_3}(s,t,0,0)$$

Suppose $J(0,0,c)$ is not invertible. Then its columns must be linearly dependent, so there must be a nontrivial relation

$$(\lambda_2 s + \mu_2 t)\frac{\partial f_c}{\partial x_2}(s,t,0,0) + (\lambda_3 s + \mu_3 t)\frac{\partial f_c}{\partial x_3}(s,t,0,0) = 0, \tag{4.8}$$

where the equality holds identically as an equality of polynomials in the variables s and t. Rearranging 4.8 reveals that $(\partial f_c/\partial x_2)(s,t,0,0)$ and $(\partial f_c/\partial x_3)(s,t,0,0)$ share a common linear factor. Therefore, there exists a point $P = [x_0 : x_1 : 0 : 0] \in \mathbb{P}^3$ such that $(\partial f_c/\partial x_2)(P) = (\partial f_c/\partial x_3)(P) = 0$.

Let L_0 be the line through $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$. This is the line with $G(1,\mathbb{P}^3)$-coodinates $(0,0)$. Since we assume $(0,0,c) \in M$, we are assuming that L_0 lies on the cubic surface X_c. Hence,

$$f_c(x_0,x_1,x_2,x_3) = x_2 g_2(x_0,x_1,x_2,x_3) + x_3 g_3(x_0,x_1,x_2,x_3), \tag{4.9}$$

for some polynomials g_2 and g_3. In particular, we have

$$\frac{\partial f_c}{\partial x_0} = x_2\frac{\partial g_2}{\partial x_0} + x_3\frac{\partial g_3}{\partial x_0}$$
$$\frac{\partial f_c}{\partial x_1} = x_2\frac{\partial g_2}{\partial x_1} + x_3\frac{\partial g_3}{\partial x_1}, \tag{4.10}$$

so it follows that $(\partial f_c/\partial x_0)(P) = (\partial f_c/\partial x_1)(P) = 0$. So P is a point such that $(\partial f_c/\partial x_i)(P) = 0$ for all $i \in \{0,1,2,3\}$. Hence by the Jacobi criterion for smoothness, P is a singular point of the cubic surface X_c. This contradicts our assumption that X_c is smooth. Therefore, the matrix $J(0,0,c)$ must be invertible.

Let $S \subseteq \mathbb{P}^{19}$ be the collection of points in the moduli space corresponding to smooth cubics. We must make the assumption that S is connected (see [8]). Nonetheless, since we have shown that $J(a,b,c)$ is invertible whenever X_c is smooth, by the implicit function theorem the coordinates a and b are determined by the c_α. Since $S \subseteq \mathbb{P}^{19}$ is connected, this means that when we modify the projection map to only consider smooth cubics: $\pi'\colon M' \to S$, then π' is a covering map on a connected space. Standard arguments in point-set topology then imply that the fibers of π' all have the same cardinality (see for instance [4]). $\square$

We can now show that the number of lines on a smooth cubic surface in $\mathbb{P}^3$ is finite. To do this, one may just compute with a simple example of a smooth cubic.

Proposition 5. *The number of lines on a smooth cubic surface in $\mathbb{P}^3$ is finite.*

Proof. Consider the smooth cubic cut out by polynomial $f = x_0^3 + x_1^3 + x_2^3 + x_3^3$, known as the *Fermat cubic.* This cubic is easily checked to be smooth by the Jacobi criterion. Upon permuting the coordinates, every line in $\mathbb{P}^3$ can be of the form $x_0 = a_2x_2 + a_3x_3$ and $x_1 = b_2x_2 + b_3x_3$. We can then these equations into f to obtain a system of equations in the a_i and b_i. Some simple algebra reveals that there are finitely many solutions. Since there are finitely many permutations of the coordinates, it follows that that there are finitely many lines on the Fermat cubic. The conclusion follows from Proposition 4. $\square$

We derived Proposition 5 by computing with a specific cubic surface. A more general approach to check the dimension hypothesis of Theorem 4 is to employ *deformation theory.* However, this lies outside of the set of tools which we have developed and we will not say more about this here.

Note that if one actually does the calculations in detail in the proof of Proposition 5, one will find the exact number of lines on any smooth cubic surface in $\mathbb{P}^3$. However, this manner of computing the numerical answer is unsatisfying. The number appears almost out of complete coincidence. In the next section, we will apply our theory of Chern classes to gain a better understanding of where the numerical answer comes from.

4.2 The Top Chern Class

We are finally in a position to answer the question we posed at the beginning of the chapter.

Theorem 5. *Every smooth cubic surface in $\mathbb{P}^3$ contains precisely twenty-seven lines.*

Proof. Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface cut out by the polynomial f. Consider the Grassmannian $G(1, \mathbb{P}^3) \cong G(2, \mathbb{A}^4)$. We may consider the tautological subbundle S on $G(1, \mathbb{P}^3)$ where the S-fiber over a point in $G(1, \mathbb{P}^3)$ is precisely that projective line itself (or alternatively the affine plane whose projectivization is the projective line itself). Note that $\operatorname{rank} S = 2$. We define the incidence correspondence by $M \coloneqq \{\Lambda \in G(1, \mathbb{P}^3) \colon \Lambda \subseteq X\}$.

Consider the vector bundle $p\colon \operatorname{Sym}^3(S^\vee) \to G(1, \mathbb{P}^3)$. This vector bundles has a section $f^\sharp$ defined by $f^\sharp([\Lambda]) \coloneqq f|_\Lambda \in \operatorname{Sym}^3(\Lambda^\vee)$. The symmetric power $\operatorname{Sym}^3(\Lambda^\vee)$ is precisely the space of homogeneous cubic polynomials on the line Λ, and the section $f^\sharp$ is the section that simply restricts the cubic polynomial f to each line $\Lambda \in G(1, \mathbb{P}^3)$. The zero locus of this section, $Z(f^\sharp)$, is precisely the lines in $\mathbb{P}^3$ on which f vanishes. That is, the lines on the cubic surface X. The degree of the divisor class we obtain will thus be the number of lines on the cubic surface X. Hence, we must simply compute $\deg Z(f^\sharp)$.

The Grassmannian $G(2, \mathbb{A}^4)$ has dimension $2 \cdot 2 = 4$ and the symmetric power $\operatorname{Sym}^3(S^\vee)$ has rank $\binom{2+3-1}{3} = 4$. By Proposition 5, the zero locus is finite and thus has dimension $\dim Z(f^\sharp) =$

$0 = \dim G(2, \mathbb{A}^4) - \operatorname{rank} \operatorname{Sym}^3(S^\vee)$. Moreover, the Grasmannian is a variety by the Plücker embedding. Hence, we satisfy the hypothesis of Theorem 4 which tells us that $[Z(f^\sharp)] = c_4(\operatorname{Sym}^3(S^\vee)) \cdot [G(1, \mathbb{P}^3)]$.

Suppose $S^\vee$ has Chern roots α and β so that $c_1(S^\vee) = \alpha + \beta$ and $c_2(S^\vee) = \alpha\beta$. As stated in our discussion in Section 3.2, $\operatorname{Sym}^3(S^\vee)$ has Chern roots 3α, $2\alpha + \beta$, $\alpha + 2\beta$, and 3β. Therefore, by the splitting principle, we may write down the total Chern class of $\operatorname{Sym}^3(S^\vee)$ as

$$c(\operatorname{Sym}^3(S^\vee)) = (\mathbf{1} + 3\alpha)(\mathbf{1} + 2\alpha + \beta)(\mathbf{1} + \alpha + 2\beta)(\mathbf{1} + 3\beta) \tag{4.11}$$

The degree 4 piece of the expansion of 4.11 is not hard to compute. We obtain:

$$\begin{aligned} c_4(\operatorname{Sym}^3(S^\vee)) &= 18\alpha^3\beta + 18\alpha\beta^3 + 45\alpha^2\beta^2 \\ &= 9\alpha\beta(2\alpha^2 + 2\beta^2 + 5\alpha\beta) \\ &= 9\alpha\beta(2(\alpha+\beta)^2 + \alpha\beta) \\ &= 9(\alpha\beta)^2 + 18\alpha\beta(\alpha+\beta)^2 \\ &= 9c_2(S^\vee)^2 + 18c_1(S^\vee)^2 c_2(S^\vee). \end{aligned} \tag{4.12}$$

Similar to our interpretation of $c_4(\operatorname{Sym}^3(S^\vee))$, we may interpret the Chern class $c_2(S^\vee)$ as encoding the lines in $\mathbb{P}^3$ contained in some hyperplane H. The class $c_2(S^\vee)^2$ thus encodes the lines in $\mathbb{P}^3$ contained in two different hyperplanes. There is only one such line of course, so the degree of $c_2(S^\vee) \cdot [G(1, \mathbb{P}^3)]$ is 1.

Note that we can interpret $S^\vee$ as the exterior power $\bigwedge^2 S^\vee$. Therefore, $c_1(S^\vee)$ encodes the collection of lines in $\mathbb{P}^3$ such that the restrictions of two generic linear equations are linearly dependent on the lines. Phrased another way, this means that $c_1(S^\vee)$ encodes the collection of lines that intersect with a given generic line. Therefore, $c_1(S^\vee)^2$ encodes the collection of lines that have an intersection with two generic lines. Then $c_1(S^\vee)^2 c_2(S^\vee)$ encodes the collection of lines that intersect with two generic lines and a generic hyperplane. Of course, there is only one such line. Hence, the degree of $c_1(S^\vee)^2 c_2(S^\vee) \cdot [G(1, \mathbb{P}^3)]$ is also 1.

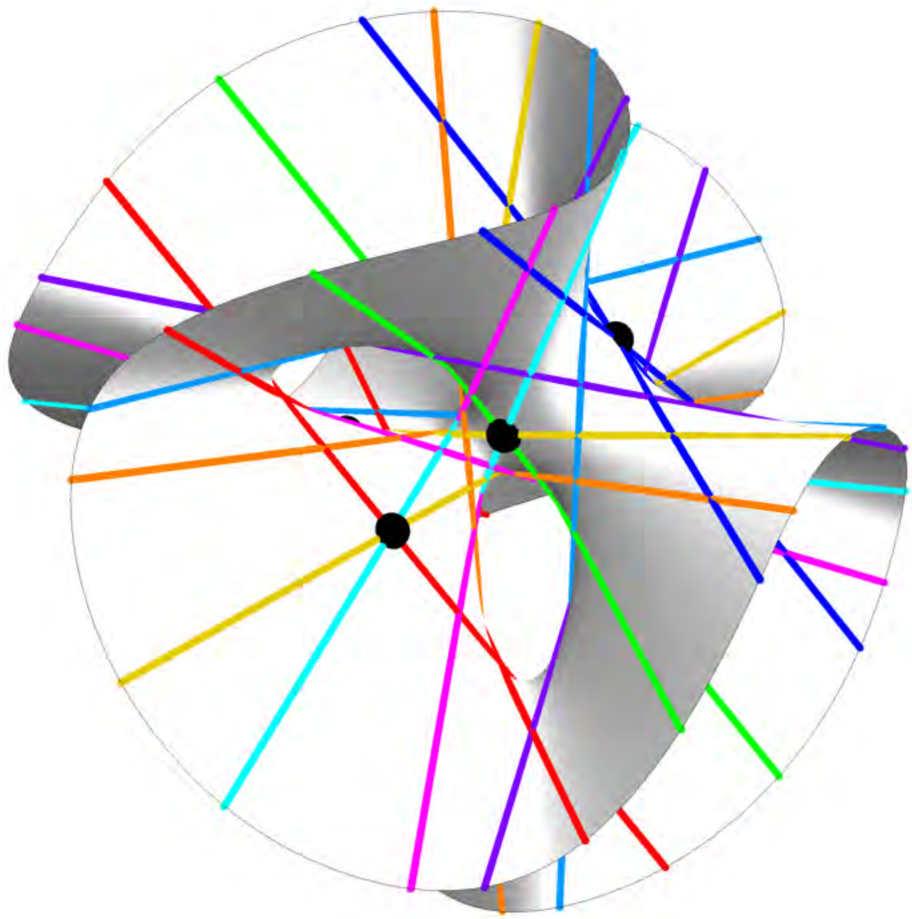
Therefore, Theorem 4 and equation 4.12 imply that the number of lines on X is

$$\deg Z(f^\sharp) = 9 + 18 = \boxed{27}. \tag{4.13}$$

□

This perspective of counting lines on a cubic surface is quite abstract. As we noted before, our work in Section 4.1 could easily be made more explicit to yield Theorem 5. However, the theory of Chern classes gives us a better perspective of *why* there are exactly 27 lines on a cubic surface. The number 27 essentially comes from the combinatorics of manipulating elementary symmetric functions of Chern roots, as we did in equation 4.12. More importantly, our approach also yields a method to solve more complicated enumerative problems.

We would be remiss if we did not conclude with a beautiful visual of the twenty-seven lines on a cubic surface ([5]).



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